

Smooth Transonic Flows in De Laval Nozzles

Chunpeng Wang*

School of Mathematics, Jilin University, Changchun 130012, P. R. China

The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong
(email: wangcp@jlu.edu.cn)

Zhouping Xin†

The Institute of Mathematical Sciences and Department of Mathematics,
The Chinese University of Hong Kong, Shatin, NT, Hong Kong
(email: zpxin@ims.cuhk.edu.hk)

Abstract

This paper concerns smooth transonic flows of Meyer type in finite de Laval nozzles, which are governed by an equation of mixed type with degeneracy and singularity at the sonic state. First we study the properties of sonic curves. For any C^2 transonic flow of Meyer type, the set of exceptional points is shown to be a closed line segment (may be empty or only one point). Furthermore, it is proved that a flow with nonexceptional points is unstable for a C^1 small perturbation in the shape of the nozzle. Then we seek smooth transonic flows of Meyer type which satisfy physical boundary conditions and whose sonic points are exceptional. For such a flow, its sonic curve must be located at the throat of the nozzle and it is strongly singular in the sense that the sonic curve is a characteristic degenerate boundary in the subsonic-sonic region, while in the sonic-supersonic region all characteristics from sonic points coincide, which are the sonic curve and never approach the supersonic region. It is proved that there exists uniquely such a smooth transonic flow near the throat of the nozzle, whose acceleration is Lipschitz continuous, if the wall of the nozzle is sufficiently flat.

Keywords: Smooth transonic flow, Equation of mixed type, Degeneracy, Singularity.

2000 MR Subject Classification: 76H05 35M12 76N10

*The research is supported by a grant from the Chinese National Sciences Foundation.

†The research is supported by Zheng Ge Ru Foundation, Hong Kong RGC Earmarked Research Grants CUHK-4011/11P and CUHK-4042/08P, a grant from the Croucher Foundation, and a Focus Area Grant from the Chinese University of Hong Kong.

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1 Introduction

In this paper we study the compressible Euler system of steady isentropic and irrotational smooth transonic flows in a class of two-dimensional de Laval nozzles. Such problems arise naturally in physical experiments and engineering designs, and there exist large literatures on experiments and numerical simulations. However, a rigorous mathematical theory remains to be completed.

A flow is called transonic if it contains both subsonic and supersonic regions, which are usually connected by shocks and sonic curves. Roughly speaking, transonic flows are governed by partial differential equations of mixed type; furthermore, shocks and sonic curves by which subsonic and supersonic regions are connected are free interfaces in general. Two kinds of transonic flow patterns have received much attention. One is transonic flows past a profile. Bers [3] showed that for two dimensional flows past an arbitrarily given profile, the whole flow field will be subsonic outside the profile if the freestream Mach number M_0 within a certain interval $0 < M_0 < \hat{M} < 1$; furthermore, the maximum flow speed will tend to the sound speed as $M_0 \rightarrow \hat{M}$. A natural question arises whether there also exists a smooth flow past the profile for $\hat{M} < M_0 < 1$. By constructing explicit solutions in the hodograph plane, one can get some smooth transonic flows for some profiles and for some values of M_0 ([4]). However, smooth transonic flows past a profile do not exist in general and are unstable even they exist according to Morawetz [16, 17, 18]. There are few studies on transonic flows with shocks past a profile and almost no rigorous results are known. The other kind is transonic flows in a nozzle. Transonic flows in nozzles were investigated first by Taylor [19] and Meyer [15], where some special solutions were shown by using power series expansions. One usually considers, as we shall do in this paper, a de Laval nozzle which is symmetric and whose cross section decreases first and then increases. There are two types of smooth transonic flows in such a nozzle, named Taylor and Meyer types. In a transonic flow of Taylor type, there are two supersonic enclosures of the type encountered in a transonic flow past a profile, and smooth transonic flows of such type do not exist in general and are unstable with respect to small changes in the shape of the nozzle even they exist ([4]). In a transonic flow of Meyer type, the sonic curve extends from one wall of the nozzle to the other, which is believed to be located near the throat of the nozzle (where the cross section is smallest). Stable transonic flows of Meyer type should be subsonic upstream and supersonic downstream, while a flow with the reversing direction is unstable with respect to small changes in the shape of the nozzle. Most known smooth transonic flows in nozzles are solved as solutions to the governing equations and no boundary conditions are discussed ([4]). So, the nozzles cannot be given in advance. It is noticed that Kuz'min [11] solved the perturbation problems of accelerating smooth transonic flows with some structural assumptions in a class of nozzles by using the principle of contracting mappings. However, the physical meaning of the boundary conditions for these smooth transonic flows is not clear. There also exists a stable discontinuous transonic flow pattern in a nozzle called transonic shocks, which are supersonic upstream and turn to subsonic across shocks. We refer to [5, 6, 7, 8, 12, 13, 14, 22] for the existence and stability of such transonic flows in flat and curved nozzles. Particularly, [12, 13, 14] solved the transonic shock pattern described by Courant and Friedrichs: Given the appropriately large receiver pressure, if the upstream flow is still supersonic behind the throat of the nozzle, then at a certain place in the divergent part of the nozzle a shock front intervenes and the gas is compressed and slowed down to subsonic speed ([9]). For these transonic shocks, the shocks are free and the flows are away from the sonic state.

The main goal of the present paper is to understand the behavior of transonic flows near the sonic state. Thus we investigate smooth transonic flows in a class of two-dimensional solid de Laval nozzles. As usual, it is assumed that the throat of the nozzle lies in the y -axis and the nozzle is symmetric

with respect to the x -axis. For convenience, we consider only the upper part of the nozzle due to its symmetry and the upper wall is given by

$$\Gamma_{\text{upw}} : y = f(x), \quad l_- \leq x \leq l_+,$$

where $l_- < 0 < l_+$ and $f \in C^1([l_-, l_+])$ satisfies

$$f'(x) \begin{cases} < 0, & l_- \leq x < 0, \\ = 0, & x = 0, \\ > 0, & 0 < x \leq l_+. \end{cases}$$

For the given inlet

$$\Gamma_{\text{in}} : x = g(y), \quad 0 \leq y \leq f(l_-),$$

we seek a smooth transonic flow of Meyer type in the nozzle whose velocity vector is along the normal direction at the inlet. Since the flow is supersonic downstream, we choose the outlet

$$\Gamma_{\text{out}} : x = t(y), \quad 0 \leq y \leq f(l_+)$$

as a free boundary where the velocity vector is along the normal of the outlet. Let Ω be the domain bounded by Γ_{upw} , the x -axis, Γ_{in} and Γ_{out} . The transonic flow in Ω satisfies the steady isentropic compressible Euler system:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad (x, y) \in \Omega, \quad (1.1)$$

$$\frac{\partial}{\partial x}(P + \rho u^2) + \frac{\partial}{\partial y}(\rho uv) = 0, \quad (x, y) \in \Omega, \quad (1.2)$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(P + \rho v^2) = 0, \quad (x, y) \in \Omega, \quad (1.3)$$

where (u, v) , P and ρ represent the velocity, pressure and density of the flow, respectively. The flow is assumed to be isentropic so that $P = P(\rho)$ is a smooth function. In particular, for a polytropic gas with the adiabatic exponent $\gamma > 1$,

$$P(\rho) = \frac{1}{\gamma} \rho^\gamma \quad (1.4)$$

is the normalized pressure. Assume further that the flow is irrotational, i.e.

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad (x, y) \in \Omega. \quad (1.5)$$

Then the density ρ is expressed in terms of the speed q according to the Bernoulli law ([9])

$$\rho(q^2) = \left(1 - \frac{\gamma-1}{2} q^2\right)^{1/(\gamma-1)}, \quad q = \sqrt{u^2 + v^2}, \quad 0 < q < q_{\max} = \left(\frac{2}{\gamma-1}\right)^{1/2}. \quad (1.6)$$

Summing up, the flow is governed by the system (1.1)–(1.6). It is well known that the system (1.1)–(1.6) can be transformed into the full potential equation

$$\operatorname{div}(\rho(|\nabla \varphi|^2) \nabla \varphi) = 0, \quad (x, y) \in \Omega, \quad (1.7)$$

where φ is the velocity potential.

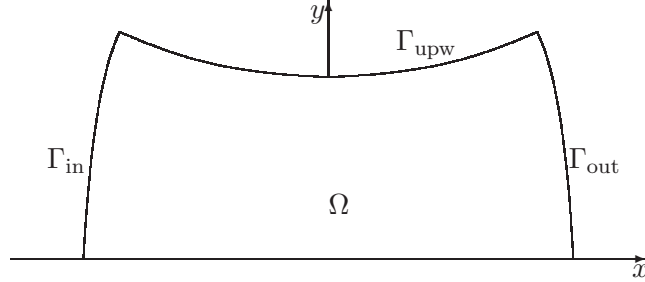


Figure: the de Laval nozzle

For a smooth transonic flow of Meyer type, the governing equation (1.7) is elliptic in the subsonic region and hyperbolic in the supersonic region. It should be noted that such a flow is singular near the sonic curve in the sense that (1.7) is degenerate elliptic in the subsonic-sonic region while non-strictly hyperbolic in the sonic-supersonic region. These factors cause the essential difficulties for the mathematical analysis. Another difficulty is that the sonic curve is usually believed to be free. So, understanding the behavior of sonic curves is not only important for physical applications and engineering designs but also one of the keys to study smooth transonic flows. Bers [2, 4] studied the continuation of a flow across a sonic curve, where a subsonic-sonic flow was assumed to be given ahead. He found out that the existence and the structure of exceptional points play an important role in this continuation. The definition of exceptional points is referred to [4] (also § 2.2). For a C^2 transonic flow, a sonic point is exceptional is equivalent to that the velocity vector is orthogonal to the sonic curve at this point. That is to say, exceptional points are characteristic degenerate in the subsonic-sonic region. If there is not any exceptional point, then, the flow can be continued in a unique way across the sonic curve as a supersonic flow without discontinuity. Whereas, if there is a unique exceptional point, the flow will be uniquely determined only in a strict subset of the same domain; furthermore, if the flow can be continued at all into this excluded region, this continuation will not be unique. Bers also mentioned in [4] that it would be interesting to know whether exceptional points are always isolated. Here, the supersonic flow is solved as a Cauchy problem from the sonic curve and there is no prescription of boundary conditions on the walls. For a smooth transonic flow of Meyer type in a nozzle, the supersonic flow satisfies not a Cauchy problem but an initial-boundary value problem. By some precise analysis on sonic curves, it is shown in the paper that for any C^2 transonic flow, the set of exceptional points is a closed line segment (may be empty or only one point). Furthermore, exceptional points are strongly singular in the sonic-supersonic region in the sense that there are two different characteristics from each nonexceptional point in the nozzle, while all characteristics from interior exceptional points coincide and they never approach the supersonic region locally. Then, it is proved that a flow with nonexceptional points is unstable for a C^1 small perturbation on the wall (even if the wall is still smooth). This instability is weak since it is unknown whether the flow is unstable if the wall is perturbed in C^2 or other smooth spaces. So we seek a smooth transonic flow of Meyer type whose sonic points are exceptional. For such a flow, its sonic curve must be located at the throat of the nozzle and the potential on its sonic curve equals identically to a constant. Another motivation arises from our early paper [20], where the structural stability problem of a symmetric

continuous subsonic-sonic flow in a convergent nozzle with straight solid walls was proved and the sonic curve of the continuous subsonic-sonic flow is shown to be a free boundary where the potential equals identically to a constant. In [20] the continuous subsonic-sonic flow is singular in the sense that while the speed is continuous yet the acceleration blows up at the sonic state. This singularity arises from the geometry of the nozzle and it seems to vanish if the sonic curve is located at the throat of a de Laval nozzle. It is noted that for a smooth transonic flow of Meyer type whose sonic curve is located at the throat of the nozzle, the potential at the sonic curve, which equals identically a constant, is free although the location of the sonic curve in the nozzle is known.

For smooth transonic flows, the governing equation is elliptic in the subsonic region and hyperbolic in the supersonic region; furthermore, it is degenerate and singular at the sonic state. So, to study a smooth transonic flow problem, one must determine and control precisely the speed of the flow near the sonic state, which plays an essential role in the mathematical analysis. In (1.7), the speed of the flow is the absolute value of the gradient of a solution. So, generally speaking, it is very hard to estimate precisely the speed of the flow near the sonic state in the physical plane even if the location of the sonic curve is known. It is more convenient to study smooth transonic flows in the potential plane, where the speed of the flow is a solution to the governing equation. After confirming the sonic curve to be a free interface where the potential equals identically to a constant, we decompose the smooth transonic flow problem into a smooth subsonic-sonic flow problem with free boundary and a smooth sonic-supersonic flow problem with fixed boundary in the potential plane, which can be solved separately. For the smooth subsonic-sonic flow problem, we encounter the two main difficulties in the study on the continuous subsonic-sonic flow problem in [20]. One is that the problem is a characteristic degenerate free boundary problem and the degeneracy occurs just at the free boundary. The other is that at the inlet we should prescribe a Neumann boundary condition instead of a Robin one with which the problem may be ill-posed. However, besides these difficulties, there is an additional disadvantage in this study. Different from the problem in [20], there are no background solutions, which play an important role to determine and control the rate of the flow from the subsonic region to the sonic state. Furthermore, the boundary conditions at the inlet and the wall are nonlinear, nonlocal and implicit. In the paper, we try to seek a subsonic-sonic flow which tends uniformly to the sonic state along different streamlines although there are no background solutions. To this end, we assume that the convergent part of the nozzle slopes so gently that

$$f''(x) = o(x^2), \quad x \rightarrow 0^-. \quad (1.8)$$

Indeed, (1.8) is necessary to guarantee that the change of the speed along the stream direction is infinitesimal of higher order than the one along the potential direction near the sonic state. By a fixed point argument with many very precise elliptic estimates, we obtain a subsonic-sonic flow which tends uniformly to the sonic state along different streamlines. For the smooth sonic-supersonic flow problem, the governing equation is a non-strictly hyperbolic equation with strong singularity at the sonic curve, where its two eigenvalues coincide and the eigenvector space reduces to a one-dimensional space. Furthermore, the singularity at the sonic curve is so strong that all characteristics from sonic points coincide, which are the sonic curve and never approach the supersonic region. And the boundary condition on the wall is nonlinear, nonlocal and implicit. As mentioned above, a crucial step to solve the problem is how to determine and control the rate of the flow from the sonic state to supersonic region. We seek a sonic-supersonic flow which moves uniformly away from the sonic state along different streamlines in the paper. For the sonic-supersonic flow in the divergent part of the de Laval nozzle,

$$f''(x) = o(x^2), \quad x \rightarrow 0^+ \quad (1.9)$$

is necessary to guarantee that the change of the speed along the stream direction is infinitesimal of higher order than the one along the potential direction near the sonic state. We solve this sonic-supersonic flow problem under the assumption (1.9) by a fixed point argument and the method of characteristics. Due to the strong singularity, the computations of the flow near the sonic curve are quite complicated. Through some precise calculations and optimal estimates, we are able to get a desired sonic-supersonic flow.

In the present paper, by solving a smooth subsonic-sonic flow problem with free boundary and a smooth sonic-supersonic flow problem with fixed boundary separately, we get a smooth transonic flow of Meyer type in the de Laval nozzle Ω , whose velocity vector is along the normal direction at the inlet and which satisfies the slip condition on the wall, under the assumptions (1.8), (1.9) and $|l_{\pm}|$ being sufficiently small. Different from the examples of smooth transonic flows of Meyer type by using power series expansions, where the exceptional point is isolated, the sonic curve of the smooth transonic flow in this paper is located at the throat and each sonic point is exceptional. Thus, this transonic flow pattern is strongly singular in the sense that the sonic curve is a characteristic degenerate boundary in the subsonic-sonic region, while in the sonic-supersonic region all characteristics from sonic points coincide, which are the sonic curve and never approach the supersonic region. It is surprising that there is a smooth transonic flow for this pattern with so strong singularity. Indeed, we get a smooth transonic flow in the sense that the acceleration is Lipschitz continuous. Furthermore, the transonic flow of this pattern is also shown to be unique. It is noted that (1.8) and (1.9) are almost necessary for a C^2 transonic flow whose sonic curve is located at the throat of the nozzle. Moreover, the assumption $|l_{\pm}|$ being sufficiently small can be relaxed. In the convergent part and the divergent part of the nozzle, we get a smooth subsonic-sonic flow and a smooth sonic-supersonic flow, respectively. Using the same methods, one can get a smooth transonic flow of Meyer type in a long de Laval nozzle if it possesses the same properties near the throat ((1.8) and (1.9), which are almost necessary) and its walls slope gently away from the throat. More generally, we believe that there also exist smooth transonic flows for general finite de Laval nozzles satisfying (1.8) and (1.9). In the upstream, by more complicated elliptic estimates than the ones in the present paper, we can get a subsonic-sonic flow for a very general convergent nozzle. In the downstream, the flow can be extended for a class of long divergent nozzles by solving an initial-boundary value problem of a strictly hyperbolic equation. These questions will be dealt with in our forthcoming studies.

The paper is arranged as follows. In §2 we analyze the structure and the location of the sonic curve for C^2 transonic flows and show the instability of transonic flows with nonexceptional points, which motives us to seek a smooth transonic flow of Meyer type whose sonic points are exceptional. We formulate the problem of smooth transonic flows and state the main results (existence and uniqueness) of the paper in §3. The smooth transonic flow problem can be decomposed into a smooth subsonic-sonic flow problem with free boundary and a smooth sonic-supersonic flow problem with fixed boundary in the potential plane, which are investigated in §4 and §5, respectively.

2 Sonic curves and instability of transonic flows with nonexceptional points

In this section, we analyze the structure and the location of sonic curves for C^2 transonic flows. And we always assume that the sonic curve belongs to both the boundary of the subsonic region and the boundary of the supersonic region.

2.1 Governing equations

Let us first rewrite the system (1.1)–(1.6) in the physical plane as the Chaplygin equations in the potential plane.

Since the conservation of momentum, (1.2) and (1.3), can be derived from (1.1), (1.4)–(1.6), the flow is just governed by the conservation of mass and the condition of irrotationality:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \quad (2.1)$$

where ρ is related to the velocity (u, v) by the Bernoulli law (1.6). The sound speed c is defined as

$$c^2 = P'(\rho) = \rho^{\gamma-1} = 1 - \frac{\gamma-1}{2}q^2.$$

At the sonic state, the speed is

$$c_* = \left(\frac{2}{\gamma+1} \right)^{1/2},$$

which is called the critical speed in the sense that the flow is subsonic ($q < c$) when $0 < q < c_*$, sonic ($q = c$) when $q = c_*$ and supersonic ($q > c$) when $c_* < q < q_{max}$.

Define a velocity potential φ and a stream function ψ , respectively, by

$$\frac{\partial \varphi}{\partial x} = u, \quad \frac{\partial \varphi}{\partial y} = v, \quad \frac{\partial \psi}{\partial x} = -\rho v, \quad \frac{\partial \psi}{\partial y} = \rho u,$$

which are

$$\frac{\partial \varphi}{\partial x} = q \cos \theta, \quad \frac{\partial \varphi}{\partial y} = q \sin \theta, \quad \frac{\partial \psi}{\partial x} = -\rho q \sin \theta, \quad \frac{\partial \psi}{\partial y} = \rho q \cos \theta$$

in terms of polar coordinators in the velocity space, where θ , which is called a flow angle, is the angle of the velocity inclination to the x -axis. Direct calculations show that the system (2.1) can be reduced to the Chaplygin equations:

$$\frac{\partial \theta}{\partial \psi} + \frac{\rho(q^2) + 2q^2 \rho'(q^2)}{q \rho^2(q^2)} \frac{\partial q}{\partial \varphi} = 0, \quad \frac{1}{q} \frac{\partial q}{\partial \psi} - \frac{1}{\rho(q^2)} \frac{\partial \theta}{\partial \varphi} = 0 \quad (2.2)$$

in the potential-stream coordinates (φ, ψ) . Note that

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} = \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \varphi}{\partial y} \frac{\partial \psi}{\partial x} = \rho q^2.$$

So the coordinates transformation between the two coordinate systems is valid at least in the absence of stagnation points. Eliminating θ from (2.2) yields the following second-order quasilinear equation

$$\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (2.3)$$

where

$$A(q) = \int_{c_*}^q \frac{\rho(s^2) + 2s^2 \rho'(s^2)}{s \rho^2(s^2)} ds, \quad B(q) = \int_{c_*}^q \frac{\rho(s^2)}{s} ds, \quad 0 < q < q_{max}.$$

Here, $B(\cdot)$ is strictly increasing in $(0, q_{max})$, while $A(\cdot)$ is strictly increasing in $(0, c_*)$ and strictly decreasing in $[c_*, q_{max})$. The inverse function of $B(\cdot)$ is denoted by $B^{-1}(\cdot)$, while the inverse functions of $A(\cdot)$ lying in $(0, c_*)$ and $[c_*, q_{max})$ are denoted by $A_-^{-1}(\cdot)$ and $A_+^{-1}(\cdot)$, respectively.

It can be checked easily that both (1.7) and (2.3) are elliptic in the subsonic region ($q < c_*$) and hyperbolic in the supersonic region ($q > c_*$), while degenerate and singular at the sonic state ($q = c_*$). Therefore, the governing equations (1.7) and (2.3) for transonic flows are both second-order mixed type quasilinear equations and are both degenerate and singular at sonic states.

2.2 Exceptional points in the physical plane

We begin with a description of sonic curves in [4]. Let \mathcal{S} be a sonic curve of a C^2 transonic flow. The positive direction on \mathcal{S} is defined by requiring that, if one moves along \mathcal{S} in this direction, the subsonic region is located on the left. Then, θ_s , the derivative of θ with respect to the arc length on \mathcal{S} , satisfies

$$\theta_s = -\frac{\sin^2 \vartheta}{c_*} \frac{\partial q}{\partial \nu}, \quad (2.4)$$

where ν is the unit normal of \mathcal{S} pointing into the supersonic region and ϑ is the angle between the velocity vector and ν .

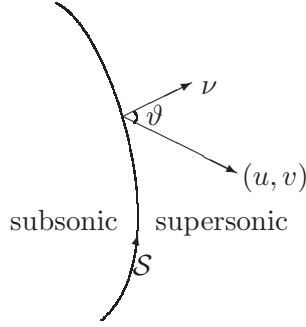


Figure: the orientation of the sonic curve

It follows from (2.4) that

Lemma 2.1 *θ is nonincreasing along \mathcal{S} .*

As in [4], points, where $\theta_s = 0$, are called exceptional. According to (2.4), a point is exceptional if and only if at this point either $\frac{\partial q}{\partial \nu} = 0$ or the velocity vector is orthogonal to the sonic curve. Indeed, at a point on \mathcal{S} , if $\frac{\partial q}{\partial \nu} = 0$, then the velocity vector is orthogonal to the sonic curve owing to the following lemma by Gilbarg and Shiffman [10].

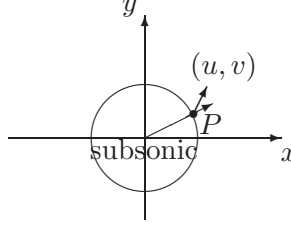


Figure of Lemma 2.2

Lemma 2.2 *Assume that $q = c_*$ at a point P on the circumference of a circle in whose interior $q < c_*$. If the velocity vector is not along the direction from the center of the circle to P , then at P the derivative of q along this direction is positive.*

Proof. The case when the circle is the interior of a subsonic-sonic region was considered in [10]. The authors derived a linear elliptic equation which is degenerate at the sonic state from (1.7). A comparison principle holds for this degenerate equation, which can be proved by the same argument for uniformly elliptic equations. Then, the lemma follows from the same proof of the Hopf lemma and the auxiliary function has been constructed in [10]. Here, it is used that the velocity vector is not along the direction from the center of the circle to P , which shows that this direction is not characteristic for the degenerate equation. The proof is standard and thus omitted. \square

Therefore, one has

Proposition 2.1 *For a C^2 transonic flow, a point at the sonic curve is exceptional if and only if the velocity vector is orthogonal to the sonic curve at this point.*

This proposition shows

Remark 2.1 *For a C^2 transonic flow, exceptional points are characteristic degenerate in the subsonic-sonic region.*

Exceptional points are regarded to be singular in the following sense ([4]). Assume that a subsonic flow is defined in some domain whose boundary contains a smooth sonic curve \mathcal{S} . If there is no exceptional point on \mathcal{S} , then, the flow can be continued in a unique way across \mathcal{S} as a supersonic flow without discontinuity. The flow will be determined in some neighborhood of \mathcal{S} contained between the two Mach lines from the two endpoints of \mathcal{S} . However, if there is a unique exceptional point on \mathcal{S} , the flow will be uniquely determined only in a subset of the same neighborhood, where the points between the two characteristics from the exceptional point are excluded; furthermore, if the flow can be continued at all into this excluded region, this continuation will not be unique. In this section, we will show that the set consisting of exceptional points for any C^2 transonic flow of Meyer type is a closed line segment (may be empty or only one point), while there is no exceptional point for any C^2 transonic flow of Taylor type. Furthermore, the singularity of exceptional points in the sonic-supersonic region is so strong that there are two different characteristics from each nonexceptional point in the nozzle, while all characteristics from interior exceptional points coincide and they never approach the supersonic region locally. In the present paper, we get a smooth transonic flow of Meyer

type whose sonic points are all exceptional. This transonic flow pattern is strongly singular in the sense that the sonic curve is a characteristic degenerate boundary in the subsonic-sonic region, while in the sonic-supersonic region all characteristics from sonic points coincide, which are the sonic curve and never approach the supersonic region. It is surprising that there is a smooth transonic flow for this pattern with so strong singularity. Indeed, we get a smooth transonic flow in the sense that the acceleration is Lipschitz continuous. This transonic flow pattern also answers the problem mentioned in [4] that it would be interesting to know whether exceptional points are always isolated.

2.3 Exceptional points in the potential plane

It turns out to be more convenient to analyze exceptional points in the potential plane. Transforming Proposition 2.1 from the Cartesian coordinates to the potential-stream coordinates, one can get

Proposition 2.2 *For a C^2 transonic flow in the potential plane, a sonic point is exceptional if and only if $\frac{\partial q}{\partial \psi} = 0$ at this point.*

Proof. On the sonic curve, it holds

$$v \frac{\partial q}{\partial x} - u \frac{\partial q}{\partial y} = v \left(u \frac{\partial q}{\partial \varphi} - \rho(c_*^2) v \frac{\partial q}{\partial \psi} \right) - u \left(v \frac{\partial q}{\partial \varphi} + \rho(c_*^2) u \frac{\partial q}{\partial \psi} \right) = -\rho(c_*^2) c_*^2 \frac{\partial q}{\partial \psi}.$$

Therefore, at a sonic point the velocity vector is orthogonal to the sonic curve in the physical plane if and only if $\frac{\partial q}{\partial \psi} = 0$ at this point. Then, the proposition follows from Proposition 2.1. \square

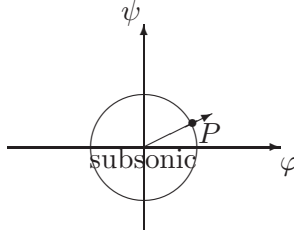


Figure of Lemma 2.3

Similar to the proof of Lemma 2.2, it can be shown that

Lemma 2.3 *Consider a C^2 flow in the potential plane. Assume that $q = c_*$ at a point P on the circumference of a circle in whose interior $q < c_*$. If the direction from the center of the circle to P is not parallel to the φ -axis, then $\frac{\partial q}{\partial \mu} > 0$ at P , where μ is any direction which forms an acute angle with the direction from the center of the circle to P .*

Proof. Without loss of generality, we assume that $P = (\varphi_0, \psi_0)$ is the unique sonic point on the circumference of the circle. Otherwise, we can consider the flow in a smaller circle which intersects the original circle at P . For convenience, it is assumed that the circle is centered at the origin with

radius r . Since the direction from the center of the circle to P is not parallel to the φ -axis, $\psi_0 \neq 0$. Set

$$\bar{q}(\varphi, \psi) = c_* - \varepsilon \left(e^{-M(\varphi^2 + \psi^2)} - e^{-Mr^2} \right), \quad (\varphi, \psi) \in \bar{B}_1 \cap \bar{B}_2$$

with positive constants $\varepsilon (\leq c_*/2)$ and M to be determined, where

$$B_1 = \{(\varphi, \psi) : \varphi^2 + \psi^2 < r^2\}, \quad B_2 = \left\{(\varphi, \psi) : (\varphi - \varphi_0)^2 + (\psi - \psi_0)^2 < \frac{1}{4}\psi_0^2\right\}.$$

Note that

$$\frac{c_*}{2} < \bar{q}(\varphi, \psi) < c_*, \quad (\varphi, \psi) \in B_1 \cap B_2$$

and

$$\begin{aligned} & \frac{\partial^2 A(\bar{q})}{\partial \varphi^2} + \frac{\partial^2 B(\bar{q})}{\partial \psi^2} \\ & \leq A'(\bar{q}(\varphi, \psi))\varepsilon(2M - 4M^2\varphi^2)e^{-M(\varphi^2 + \psi^2)} + B'(\bar{q}(\varphi, \psi))\varepsilon(2M - 4M^2\psi^2)e^{-M(\varphi^2 + \psi^2)} \\ & \leq -M\varepsilon(B'(\bar{q}(\varphi, \psi))(M\psi_0^2 - 2) - 2A'(\bar{q}(\varphi, \psi)))e^{-M(\varphi^2 + \psi^2)} \\ & \leq -M\varepsilon\left((M\psi_0^2 - 2) \min_{c_*/2 \leq s \leq c_*} B'(s) - 2 \max_{c_*/2 \leq s \leq c_*} A'(s)\right)e^{-M(\varphi^2 + \psi^2)}, \quad (\varphi, \psi) \in B_1 \cap B_2. \end{aligned}$$

Therefore, there exists a sufficiently large $M > 0$ such that \bar{q} is a supersolution to (2.3) in $B_1 \cap B_2$ for each $\varepsilon \in (0, c_*/2]$. Choose $\varepsilon \in (0, c_*/2]$ so small that

$$q(\varphi, \psi) \leq \bar{q}(\varphi, \psi), \quad (\varphi, \psi) \in B_1 \cap \partial B_2.$$

Then, it follows from a comparison principle that

$$q(\varphi, \psi) \leq \bar{q}(\varphi, \psi), \quad (\varphi, \psi) \in B_1 \cap B_2,$$

which leads to

$$\frac{\partial q}{\partial \mu}(\varphi_0, \psi_0) \geq \frac{\partial \bar{q}}{\partial \mu}(\varphi_0, \psi_0) > 0.$$

Here, the comparison principle is not the classical one since (2.3) is degenerate at the sonic state and it can be proved similar to Proposition 3.1 in [20]. \square

As a corollary of Proposition 2.2 and Lemma 2.3, it holds that

Lemma 2.4 *For a C^2 transonic flow in the potential plane, if an interior point in the nozzle is exceptional, then the sonic curve near this point is a graph of function with respect to ψ .*

Proof. If the lemma is false, then $\frac{\partial q}{\partial \varphi} = 0$ at this exceptional point. Note that Proposition 2.2 implies $\frac{\partial q}{\partial \psi} = 0$ at this exceptional point. So the derivative of q at this exceptional point along any direction is zero. Since this exceptional point is an interior point in the nozzle, Lemma 2.3 shows that the normal of the sonic curve at this exceptional point is parallel to the φ -axis, which yields that the sonic curve near this point is a graph of function with respect to ψ . \square

If an exceptional point is on the wall of the nozzle, one can get that

Lemma 2.5 Consider a C^2 transonic flow in the potential plane with the sonic curve given by

$$S : \varphi = S_1(s), \psi = S_2(s), (S_1'(s))^2 + (S_2'(s))^2 > 0, \quad 0 \leq s \leq 1, \quad S_1, S_2 \in C^2([0, 1]).$$

Assume that $S_2(0) = \inf_{(0,1)} S_2 < \sup_{(0,1)} S_2 = S_2(1)$ and the subsonic region is located on the left of S .

(i) If $S_1'(0) \geq 0$ and $(S_1(0), S_2(0))$ is exceptional, then S near $(S_1(0), S_2(0))$ is a graph of function with respect to ψ .

(ii) If $S_1'(1) \leq 0$ and $(S_1(1), S_2(1))$ is exceptional, then S near $(S_1(1), S_2(1))$ is a graph of function with respect to ψ .

Proof. We will prove (ii) only since (i) can be dealt with similarly. If $S_1'(1) = 0$, then $S_2'(1) \neq 0$ and the conclusion of the lemma is obvious. Therefore, if (ii) is false, then $S_1'(1) < 0$ and S near $(S_1(1), S_2(1))$ is a graph of function with respect to φ , which is denoted by

$$\psi = h(\varphi), \quad S_1(1) \leq \varphi \leq S_1(1) + \tau \quad (\tau > 0)$$

with

$$h'(S_1(1)) = 0. \tag{2.5}$$

It follows from $S_2(1) = \sup_{(0,1)} S_2$ that

$$h(\varphi) \leq h(S_1(1)), \quad S_1(1) \leq \varphi \leq S_1(1) + \tau,$$

which, together with (2.5), leads to

$$-\infty \leq h''(S_1(1)) \leq 0.$$

If $-\infty < h''(S_1(1)) \leq 0$, then there exists a circle whose interior is located at the subsonic region and which intersects with S at $(S_1(1), S_2(1))$. Thus Lemma 2.3 shows $\frac{\partial q}{\partial \psi}(S_1(1), S_2(1)) > 0$, which contradicts $(S_1(1), S_2(1))$ is exceptional by Proposition 2.2. Therefore,

$$h''(S_1(1)) = -\infty. \tag{2.6}$$

Due to (2.5) and (2.6), there exists a number $\varepsilon \in (0, \tau]$ such that

$$h'(\varphi) < 0, \quad S_1(1) < \varphi < S_1(1) + \varepsilon,$$

which shows that S near $(S_1(1), S_2(1))$ is a graph of function with respect to ψ . \square

Remark 2.2 As will be shown later (see Remark 2.5), neither $S_1'(0) > 0$ in (i) nor $S_1'(1) < 0$ in (ii) can occur.

Remark 2.3 In Lemma 2.5, if either $S_1'(0) < 0$ in (i) or $S_1'(1) > 0$ in (ii), then the similar proof is invalid and it is unknown whether the conclusion of the lemma holds. At this moment these two cases cannot be excluded for transonic flows in a nozzle.

2.4 Behavior of sonic curves near exceptional points

Consider a C^2 transonic flow with the sonic curve S in the potential plane. To study the behavior of the sonic curve near exceptional points, we start with a geometric property of the sonic curve.

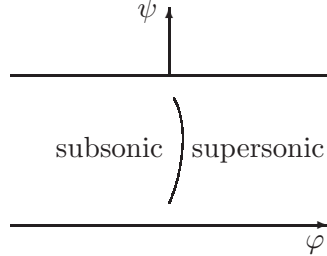


Figure of Lemma 2.6 (i)

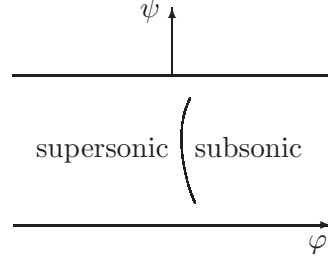


Figure of Lemma 2.6 (ii)

Lemma 2.6 Assume that $S_0 : \varphi = h_0(\psi)$ ($\psi_1 \leq \psi \leq \psi_2$, $\psi_1 < \psi_2$) is a portion of S .

(i) If the flow is subsonic in $\{(\varphi, \psi) : \mu_1 \leq \varphi < h_0(\psi), \psi_1 \leq \psi \leq \psi_2\}$ with a number $\mu_1 < \min_{[\psi_1, \psi_2]} h_0$, then

$$h_0(\psi) \geq \min\{h_0(\psi_1), h_0(\psi_2)\}, \quad \psi_1 \leq \psi \leq \psi_2.$$

(ii) If the flow is subsonic in $\{(\varphi, \psi) : h_0(\psi) < \varphi \leq \mu_2, \psi_1 \leq \psi \leq \psi_2\}$ with a number $\mu_2 > \max_{[\psi_1, \psi_2]} h_0$, then

$$h_0(\psi) \leq \max\{h_0(\psi_1), h_0(\psi_2)\}, \quad \psi_1 \leq \psi \leq \psi_2.$$

Proof. We prove (i) only and the proof of (ii) is similar. Otherwise, there exists $\psi_0 \in (\psi_1, \psi_2)$ such that

$$h_0(\psi_0) = \min_{[\psi_1, \psi_2]} h < \min\{h_0(\psi_1), h_0(\psi_2)\}.$$

Set

$$\bar{q}(\varphi, \psi) = A^{-1}(\delta(\varphi - h_0(\psi_0))), \quad \mu_1 \leq \varphi \leq h_0(\psi_0), \quad \psi_1 \leq \psi \leq \psi_2,$$

where δ is a positive number so small that

$$\sup_{(\psi_1, \psi_2)} A(q(\mu_1, \cdot)) \leq -\delta(h_0(\psi_0) - \mu_1), \quad \sup_{(\mu_1, h_0(\psi_0))} \{A(q(\cdot, \psi_1)), A(q(\cdot, \psi_2))\} \leq -\delta(h_0(\psi_0) - \mu_1).$$

Then, it follows from a comparison principle that

$$q(\varphi, \psi) \leq \bar{q}(\varphi, \psi) = A^{-1}(\delta(\varphi - h_0(\psi_0))), \quad \mu_1 \leq \varphi \leq h_0(\psi_0), \quad \psi_1 \leq \psi \leq \psi_2,$$

which leads to $\frac{\partial q}{\partial \varphi}(h_0(\psi_0), \psi_0) = +\infty$. Here, the comparison principle is not the classical one since (2.3) is degenerate at the sonic state and it can be proved similar to Proposition 3.1 in [20]. \square

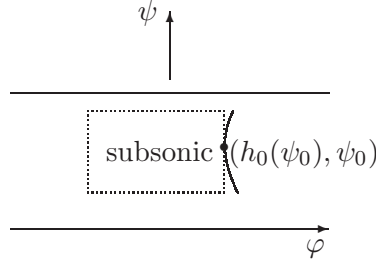


Figure of the proof of Lemma 2.6

The following lemma and proposition describe the behavior of the sonic curve near an exceptional point.

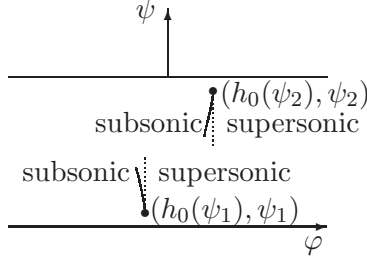


Figure of Lemma 2.7

Lemma 2.7 *Assume that $S_0 : \varphi = h_0(\psi)$ ($\psi_1 \leq \psi \leq \psi_2$, $\psi_1 < \psi_2$) is a portion of S and the flow is subsonic on the left of S_0 .*

(i) *If $\frac{\partial q}{\partial \psi}(h_0(\psi_1), \psi_1) = 0$ and $h'_0(\psi_1) = 0$, then there exists a positive constant ε ($\leq \psi_2 - \psi_1$) such that*

$$h_0(\psi) \leq h_0(\psi_1), \quad \psi_1 \leq \psi \leq \psi_1 + \varepsilon.$$

(ii) *If $\frac{\partial q}{\partial \psi}(h_0(\psi_2), \psi_2) = 0$ and $h'_0(\psi_2) = 0$, then there exists a positive constant ε ($\leq \psi_2 - \psi_1$) such that*

$$h_0(\psi) \leq h_0(\psi_2), \quad \psi_2 - \varepsilon \leq \psi \leq \psi_2.$$

Proof. We prove (ii) only and (i) can be proved similarly. Otherwise, Lemma 2.6 shows that there exists $\psi_0 \in [\psi_1, \psi_2)$ such that

$$h_0(\psi) > h_0(\psi_2), \quad h'_0(\psi) \leq 0, \quad \psi_0 < \psi < \psi_2. \quad (2.7)$$

Furthermore, (2.7), Proposition 2.2 and Lemma 2.3 lead to

$$\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) \geq 0, \quad \psi_0 \leq \psi \leq \psi_2 \quad (2.8)$$

and

$$\left\{ \psi \in (\psi_0, \psi_2) : \frac{\partial q}{\partial \psi}(h_0(\psi), \psi) > 0 \right\} \text{ is dense in any left neighborhood of } \psi_2. \quad (2.9)$$

$$B'(c_*) \frac{\partial^2 q}{\partial \psi^2}(h_0(\psi), \psi) = -A''(c_*) \left(\frac{\partial q}{\partial \varphi}(h_0(\psi), \psi) \right)^2 - B''(c_*) \left(\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) \right)^2 \geq 0, \quad \psi_0 \leq \psi \leq \psi_2. \quad (2.10)$$
$$\frac{\partial q}{\partial \varphi}(h_0(\psi_2), \psi_2) = 0. \quad (2.11)$$
$$\left. \frac{d}{d\psi} \left(\frac{\partial q}{\partial \psi} (h_0(\psi), \psi) \right) \right|_{\psi=\psi_2} = \frac{\partial^2 q}{\partial \varphi \partial \psi} (h_0(\psi_2), \psi_2) h'_0(\psi_2) + \frac{\partial^2 q}{\partial \psi^2} (h_0(\psi_2), \psi_2) > 0,$$

The diagram shows a coordinate system with a horizontal axis labeled φ and a vertical axis labeled ψ . A solid curve, labeled h_0 , separates the 'subsonic' region (to the left) from the 'supersonic' region (to the right). A dashed curve, labeled $h_0 - \mu$, is shown to the left of h_0 . The region between these two curves is labeled G . A point on the solid curve h_0 is marked with a dot and labeled $(h_0(\psi_2), \psi_2)$. Dotted lines extend from this point to the ψ -axis and to the dashed curve $h_0 - \mu$.

Assume that $\mu > 0$, $0 < \tau \leq \psi_2 - \psi_0$ and $h \in C^2([\psi_2 - \tau, \psi_2])$ satisfy

$$q(\varphi, \psi) < c_*, \quad h_0(\psi) - \mu \leq \varphi \leq h(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2, \quad (2.13)$$

$$\frac{\partial q}{\partial \psi}(\varphi, \psi_2 - \tau) > 0, \quad h_0(\psi_2 - \tau) - \mu \leq \varphi \leq h_0(\psi_2 - \tau). \quad (2.14)$$

$$\bar{q}(\varphi, \psi) = c_* + \delta(\psi - \psi_2)(h(\psi) - \varphi), \quad (\varphi, \psi) \in \overline{G},$$
$$q(h_0(\psi) - \mu, \psi) \leq \bar{q}(h_0(\psi) - \mu, \psi), \quad \psi_2 - \tau < \psi < \psi_2 \quad (2.15)$$
$$\frac{\partial q}{\partial \psi}(\varphi, \psi_2 - \tau) \geq \frac{\partial \bar{q}}{\partial \psi}(\varphi, \psi_2 - \tau), \quad h_0(\psi_2 - \tau) - \mu < \varphi < h(\psi_2 - \tau). \quad (2.16)$$

Noting

$$\frac{\partial^2 \bar{q}}{\partial \varphi^2}(\varphi, \psi) = 0, \quad \frac{\partial^2 \bar{q}}{\partial \psi^2}(\varphi, \psi) = \delta(2h'(\psi) + (\psi - \psi_2)h''(\psi)) \leq 0, \quad (\varphi, \psi) \in G$$

due to (2.12), one gets that \bar{q} is a supersolution to (2.3) in G . Since q is strictly subsonic in G from (2.13), the classical comparison principle holds. Then, it follows from (2.15) and (2.16) that

$$q(\varphi, \psi) \leq c_* + \delta(\psi - \psi_2)(h(\psi) - \varphi), \quad h_0(\psi) - \mu \leq \varphi \leq h(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2. \quad (2.17)$$

We now derive a contradiction. Owing to (2.7)–(2.9), there exist $\mu > 0$, $0 < \tau \leq \psi_2 - \psi_0$, $\{h_k\}_{k=1}^\infty \subset C^2([\psi_2 - \tau, \psi_2])$ and $\{\tau_n\}_{n=1}^\infty \subset (0, \tau)$ satisfying

$$h_0(\psi) - \mu < h_k(\psi) \leq h_{k+1}(\psi) < h_0(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2, \quad k = 1, 2, \dots, \quad (2.18)$$

$$h'_k(\psi_2) = 0, \quad h''_k(\psi) \geq 0, \quad \psi_2 - \tau \leq \psi \leq \psi_2, \quad k = 1, 2, \dots, \quad (2.19)$$

$$q(\varphi, \psi) < c_*, \quad h_0(\psi) - \mu \leq \varphi < h_0(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2, \quad (2.20)$$

$$\frac{\partial q}{\partial \psi}(\varphi, \psi_2 - \tau) > 0, \quad h_0(\psi_2 - \tau) - \mu \leq \varphi \leq h_0(\psi_2 - \tau), \quad (2.21)$$

$$\lim_{n \rightarrow \infty} \tau_n = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} h_k(\psi_2 - \tau_n) = h_0(\psi_2 - \tau_n) \text{ for } n = 1, 2, \dots. \quad (2.22)$$

Note that (2.18)–(2.21) imply that h_k satisfies (2.12)–(2.14) for each positive integer k . It follows from (2.17) that

$$q(\varphi, \psi) \leq c_* + \delta(\psi - \psi_2)(h_k(\psi) - \varphi), \quad h_0(\psi) - \mu \leq \varphi \leq h_k(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2, \quad k = 1, 2, \dots,$$

where $\delta > 0$ is independent of k . This, together with (2.22) and (2.11), leads to

$$\frac{\partial q}{\partial \varphi}(h_0(\psi_2 - \tau_n), \psi_2 - \tau_n) \geq \delta \tau_n, \quad n = 1, 2, \dots$$

and thus

$$\frac{\partial^2 q}{\partial \varphi \partial \psi}(h_0(\psi_2), \psi_2) \leq -\delta. \quad (2.23)$$

Due to (2.10) and (2.23), there exists a number $\varepsilon \in (0, \tau)$ such that

$$\frac{d}{d\psi} \left(\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) \right) = \frac{\partial^2 q}{\partial \varphi \partial \psi}(h_0(\psi), \psi) h'_0(\psi) + \frac{\partial^2 q}{\partial \psi^2}(h_0(\psi), \psi) \geq -\frac{\delta}{2} h'_0(\psi), \quad \psi_2 - \varepsilon < \psi < \psi_2,$$

which leads to

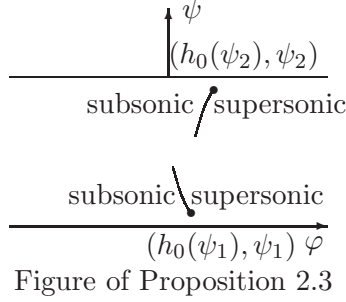
$$\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) - \frac{\partial q}{\partial \psi}(h_0(\psi_2), \psi_2) \leq -\frac{\delta}{2}(h_0(\psi) - h_0(\psi_2)), \quad \psi_2 - \varepsilon \leq \psi \leq \psi_2.$$

This, together with $\frac{\partial q}{\partial \psi}(h_0(\psi_2), \psi_2) = 0$, (2.7) and (2.8), yields that

$$\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) = 0, \quad \psi_2 - \varepsilon \leq \psi \leq \psi_2,$$

which contradicts (2.9). □

Let us remove the restrictions $h'_0(\psi_1) = 0$ and $h'_0(\psi_2) = 0$ in Lemma 2.7.



Proposition 2.3 *Assume that $S_0 : \varphi = h_0(\psi)$ ($\psi_1 \leq \psi \leq \psi_2$, $\psi_1 < \psi_2$) is a portion of S and the flow is subsonic on the left of S_0 .*

(i) *If $\frac{\partial q}{\partial \psi}(h_0(\psi_1), \psi_1) = 0$, then there exists a positive constant $\varepsilon (\leq \psi_2 - \psi_1)$ such that*

$$h_0(\psi) \leq h_0(\psi_1), \quad \psi_1 \leq \psi \leq \psi_1 + \varepsilon.$$

(ii) *If $\frac{\partial q}{\partial \psi}(h_0(\psi_2), \psi_2) = 0$, then there exists a positive constant $\varepsilon (\leq \psi_2 - \psi_1)$ such that*

$$h_0(\psi) \leq h_0(\psi_2), \quad \psi_2 - \varepsilon \leq \psi \leq \psi_2.$$

Proof. We prove (ii) only and the proof of (i) is similar. If the lemma is false, then Lemma 2.7 implies that $-\infty \leq h'_0(\psi_2) < 0$ and thus

$$\frac{\partial q}{\partial \varphi}(h_0(\psi_2), \psi_2) = 0. \quad (2.24)$$

Since the flow is subsonic on the left of $(h_0(\psi_2), \psi_2)$, (2.24) implies

$$\frac{\partial^2 q}{\partial \varphi^2}(h_0(\psi_2), \psi_2) \leq 0. \quad (2.25)$$

Choose a positive number $\tau (< \psi_2 - \psi_1)$ such that

$$h'_0(\psi) < 0, \quad \psi_2 - \tau \leq \psi < \psi_2. \quad (2.26)$$

It follows from Lemma 2.3 and (2.26) that $\frac{\partial q}{\partial \psi}(h_0(\psi_2 - \tau), \psi_2 - \tau) > 0$. Let μ be a positive number so small that $\{(\varphi, \psi) : h_0(\psi) - \mu \leq \varphi < h_0(\psi), \psi_2 - \tau < \psi < \psi_2\}$ belongs to the subsonic region and

$$\frac{\partial q}{\partial \psi}(\varphi, \psi_2 - \tau) \geq \frac{1}{2} \frac{\partial q}{\partial \psi}(h_0(\psi_2 - \tau), \psi_2 - \tau), \quad h_0(\psi_2 - \tau) - \mu < \varphi < h_0(\psi_2 - \tau).$$

We will use a similar method as in Lemma 2.7 to complete the proof. Set

$$\mathcal{D} = \left\{ h \in C^2([\psi_2 - \tau, \psi_2]) : h_0(\psi) - \mu < h(\psi) < h_0(\psi), h'(\psi) < 0 \right\}$$

and $(\psi_2 - \psi)h''(\psi) \geq 2h'(\psi)$ for each $\psi_2 - \tau \leq \psi \leq \psi_2$ }.

For any $0 < \varepsilon < \tau$, if $h \in C^2([\psi_2 - \varepsilon, \psi_2])$ satisfies

$$(\psi_2 - \psi)h''(\psi) \leq 2h'(\psi) < 0, \quad \psi_2 - \varepsilon \leq \psi < \psi_2,$$

then h must be unbounded. Hence, there exist $\{h_k\}_{k=1}^\infty \subset \mathcal{D}$ and $\{\tau_n\}_{n=1}^\infty \subset (0, \tau)$ such that

$$\lim_{n \rightarrow \infty} \tau_n = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} h_k(\psi_2 - \tau_n) = h_0(\psi_2 - \tau_n) \text{ for } n = 1, 2, \dots. \quad (2.27)$$

For each $k \geq 1$, set

$$\bar{q}_k(\varphi, \psi) = c_* + \delta(\psi - \psi_2)(h_k(\psi) - \varphi), \quad h_0(\psi) - \mu \leq \varphi \leq h_k(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2,$$

where δ , which is independent of k , is a positive number so small that

$$q(h_0(\psi) - \mu, \psi) \leq \bar{q}_k(h_0(\psi) - \mu, \psi), \quad \psi_2 - \tau < \psi < \psi_2 \quad (2.28)$$

and

$$\frac{\partial q}{\partial \psi}(\varphi, \psi_2 - \tau) \geq \frac{\partial \bar{q}_k}{\partial \psi}(\varphi, \psi_2 - \tau), \quad h_0(\psi_2 - \tau) - \mu < \varphi < h_k(\psi_2 - \tau). \quad (2.29)$$

Note that

$$\frac{\partial^2 \bar{q}_k}{\partial \psi^2}(\varphi, \psi) = \delta(2h'_k(\psi) + (\psi - \psi_2)h''_k(\psi)) \leq 0, \quad h_0(\psi) - \mu < \varphi < h_k(\psi), \quad \psi_2 - \tau < \psi < \psi_2$$

due to the definition of \mathcal{D} . Thus \bar{q}_k is a supersolution to (2.3). Then one can get from the classical comparison principle, together with (2.28) and (2.29), that

$$q(\varphi, \psi) \leq c_* + \delta(\psi - \psi_2)(h_k(\psi) - \varphi), \quad h_0(\psi) - \mu \leq \varphi \leq h_k(\psi), \quad \psi_2 - \tau \leq \psi \leq \psi_2, \quad k = 1, 2, \dots,$$

which, together with (2.27), yields

$$\frac{\partial q}{\partial \varphi}(h_0(\psi_2 - \tau_n), \psi_2 - \tau_n) \geq \delta \tau_n, \quad n = 1, 2, \dots. \quad (2.30)$$

Then (2.30), (2.24), (2.25) and $-\infty \leq h'_0(\psi_2) < 0$ imply that

$$\begin{aligned} -\delta &\geq \frac{d}{d\psi} \left(\frac{\partial q}{\partial \varphi}(h_0(\psi), \psi) \right) \Big|_{\psi=\psi_2} = \frac{\partial^2 q}{\partial \varphi^2}(h_0(\psi_2), \psi_2) h'_0(\psi_2) + \frac{\partial^2 q}{\partial \varphi \partial \psi}(h_0(\psi_2), \psi_2) \\ &\geq \frac{\partial^2 q}{\partial \varphi \partial \psi}(h_0(\psi_2), \psi_2). \end{aligned} \quad (2.31)$$

Additionally, (2.3) yields

$$B'(c_*) \frac{\partial^2 q}{\partial \psi^2}(h_0(\psi_2), \psi_2) = -A''(c_*) \left(\frac{\partial q}{\partial \varphi}(h_0(\psi_2), \psi_2) \right)^2 - B''(c_*) \left(\frac{\partial q}{\partial \psi}(h_0(\psi_2), \psi_2) \right)^2 \geq 0. \quad (2.32)$$

It follows from (2.31), (2.32) and $-\infty \leq h'_0(\psi_2) < 0$ that

$$\frac{d}{d\psi} \left(\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) \right) \Big|_{\psi=\psi_2} = \frac{\partial^2 q}{\partial \varphi \partial \psi}(h_0(\psi_2), \psi_2) h'_0(\psi_2) + \frac{\partial^2 q}{\partial \psi^2}(h_0(\psi_2), \psi_2) \in (0, +\infty]. \quad (2.33)$$

Due to (2.33) and $\frac{\partial q}{\partial \psi}(h_0(\psi_2), \psi_2) = 0$, one can get that for sufficiently small $\varepsilon \in (0, \tau)$,

$$\frac{\partial q}{\partial \psi}(h_0(\psi), \psi) < 0, \quad \psi_2 - \varepsilon < \psi < \psi_2,$$

which contradicts (2.26) and q is subsonic on the left of S_0 . \square

According to Lemmas 2.4, 2.5 and Proposition 2.3, one can get the structure of exceptional points.

Remark 2.4 Consider a C^2 transonic flow in the potential plane. If an interior point in the nozzle is exceptional, then near the exceptional point, the sonic curve can be represented as $\varphi = h(\psi)$ which achieves a local maximum (if the left of this point is subsonic) or minimum (if the right of this point is subsonic) at this point.

Remark 2.5 Under the assumptions of Lemma 2.5, one further gets $S'_1(0) = 0$ in (i) and $S'_1(1) = 0$ in (ii).

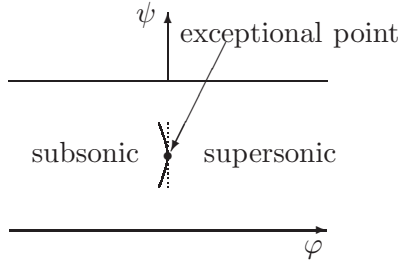


Figure of Remark 2.4

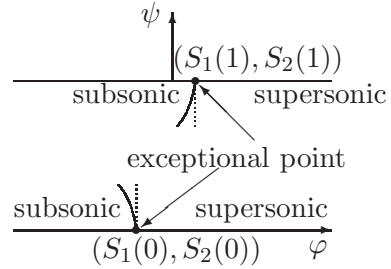


Figure of Remark 2.5

Remark 2.6 We never use supersonic regions in the discussion in § 2.2, 2.3, 2.4 except the orientation of sonic curves. So, the results in these three subsections (except Lemma 2.1) also hold for any C^2 subsonic-sonic flow or subsonic-sonic-subsonic flow whose sonic points is a curve.

2.5 Structure of sonic curves

Consider a nozzle in the potential plane of the form

$$G = \{(\varphi, \psi) : g_1(\psi) < \varphi < g_2(\psi), \psi_1 < \psi < \psi_2\},$$

where $\psi_1 < \psi_2$ and

$$g_1(\psi) < g_2(\psi), \quad \psi_1 \leq \psi \leq \psi_2.$$

The structure of the sonic curve of a C^2 transonic flow in G is as follows.

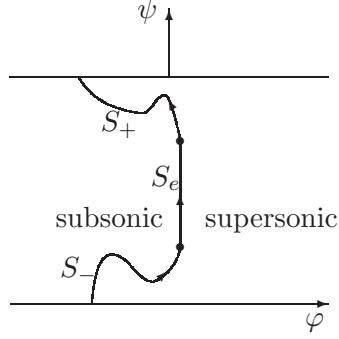


Figure of Theorem 2.1

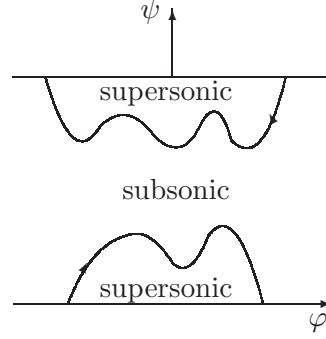


Figure of Theorem 2.2

Theorem 2.1 *Let S be the sonic curve of a $C^2(\overline{G})$ transonic flow of Meyer type. Then, S is a disjoint union of three connected parts (may be empty): S_e , S_+ , S_- , where S_e is the set of exceptional points, while S_+ and S_- denote the other two connected parts approaching the upper and lower walls, respectively. Furthermore,*

- (i) S_e is a closed segment parallel to the ψ -axis;
- (ii) S_+ and S_- are two graphes of function with respect to φ , respectively. Along the positive direction of S , φ is strictly decreasing on S_+ while strictly increasing on S_- .
- (iii) If S_e is empty, then $S = S_+$ or $S = S_-$.

Proof. Denote

$$S : \varphi = S_1(s), \psi = S_2(s), (S_1'(s))^2 + (S_2'(s))^2 > 0, \quad 0 \leq s \leq 1, \quad S_1, S_2 \in C^2([0, 1]).$$

Without loss of generality, it is assumed that $S_2(0) = \psi_1$ and $S_2(1) = \psi_2$, which is equivalent to that the subsonic region is located on the left of S . Note

$$\frac{\partial q}{\partial \varphi}(S_1(s), S_2(s))S_1'(s) + \frac{\partial q}{\partial \psi}(S_1(s), S_2(s))S_2'(s) = 0, \quad 0 \leq s \leq 1.$$

It is clear that

$$\text{if } S_1'(s) = 0 \text{ for some } s \in [0, 1], \text{ then } \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) = 0. \quad (2.34)$$

Lemma 2.4 and Proposition 2.3 show that

$$\text{if } \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) = 0 \text{ for some } s \in (0, 1), \text{ then } S_1'(s) = 0. \quad (2.35)$$

Owing to (2.34) and Proposition 2.2, $S_1'(0) \neq 0$ if $(S_1(0), S_2(0))$ is not exceptional. There are four cases to be considered.

- (i) The case that $(S_1(0), S_2(0))$ is exceptional and $S_1'(0) \geq 0$. Then, it follows from Propositions 2.2, 2.3 and Lemma 2.5 that $S_1'(0) = 0$. Set

$$s^* = \sup\{0 \leq s \leq 1 : S_1'(s) = 0\}.$$

Due to (2.34) and Lemmas 2.6, 2.7, one gets that

$$S_1'(s) = 0, \quad 0 \leq s \leq s^*. \quad (2.36)$$

Indeed, if (2.36) is false, there exist $s_1, s_2 \in [0, s^*]$ such that $s_1 < s_2$, $S'_1(s_1) = S'_1(s_2) = 0$ and

$$S'_1(s) \neq 0, \quad s_1 < s < s_2. \quad (2.37)$$

It follows from (2.34) and Lemma 2.7 that

$$S_1(s) \leq S_1(s_1), \quad s_1 \leq s \leq s_1 + \varepsilon, \quad (2.38)$$

$$S_1(s) \leq S_1(s_2), \quad s_2 - \varepsilon \leq s \leq s_2 \quad (2.39)$$

for some positive number $\varepsilon (\leq s_2 - s_1)$. Additionally, Lemma 2.6 yields

$$S_1(s) \geq \min\{S_1(s_1), S_1(s_2)\}, \quad s_1 \leq s \leq s_2. \quad (2.40)$$

Due to (2.38)–(2.40), $S'_1 = 0$ either on $[s_1, s_1 + \varepsilon]$ or on $[s_2 - \varepsilon, s_2]$, which contradicts (2.37). Owing to (2.36), (2.34) and Proposition 2.2, $(S_1(s), S_2(s))$ is exceptional for each $s \in [0, s^*]$. If $s^* = 1$, $S_e = S$ is a closed segment parallel to the ψ -axis. Otherwise, $0 \leq s^* < 1$. Lemma 2.7 and the definition of s^* imply

$$S'_1(s) < 0, \quad s^* < s \leq 1.$$

It follows from Proposition 2.2 and (2.35) that $(S_1(s), S_2(s))$ is not exceptional for each $s \in (s^*, 1)$. Furthermore, Propositions 2.2, 2.3 and Lemma 2.5 show that $(S_1(1), S_2(1))$ is not exceptional either. That is to say, $S_e = \{(S_1(s), S_2(s)) : 0 \leq s \leq s^*\}$ and $S_+ = \{(S_1(s), S_2(s)) : s^* < s \leq 1\}$.

(ii) The case that $(S_1(0), S_2(0))$ is exceptional and $S'_1(0) < 0$. Let us prove that $S'_1(s) < 0$ for each $s \in [0, 1]$ by contradiction. Otherwise, there exists $s_0 \in (0, 1]$ such that $S'_1(s_0) = 0$ and $S'_1(s) < 0$ for each $s \in [0, s_0)$. Then, (2.34) and Lemma 2.7 show that $S_1(s_0 - \varepsilon) \leq S_1(s_0)$ for sufficiently small $\varepsilon \in (0, s_0)$, which contradicts that $S'_1(s) < 0$ for each $s \in [0, s_0)$. Therefore,

$$S'_1(s) < 0, \quad 0 \leq s \leq 1.$$

It follows from Proposition 2.2 and (2.35) that $(S_1(s), S_2(s))$ is not exceptional for each $s \in (0, 1)$. Furthermore, $(S_1(1), S_2(1))$ is not exceptional either due to Propositions 2.2, 2.3 and Lemma 2.5. That is to say, $S_e = \{(S_1(0), S_2(0))\}$ and $S_+ = \{(S_1(s), S_2(s)) : 0 < s \leq 1\}$.

(iii) The case that $(S_1(0), S_2(0))$ is not exceptional and $S'_1(0) > 0$. Set

$$s_* = \sup\{0 \leq s \leq 1 : S'_1(s) > 0\}.$$

Then $0 < s_* \leq 1$. Let us prove

$$S'_1(s) > 0, \quad 0 \leq s < s_* \quad (2.41)$$

by contradiction. Otherwise, there exist $s_0 \in (0, s_*)$ and $\varepsilon_0 \in (0, s_* - s_0]$ such that $S'_1(s_0) = 0$ and $S'_1(s) > 0$ for each $s \in (s_0, s_0 + \varepsilon_0)$. Then, (2.34) and Lemma 2.7 show that $S_1(s_0 + \varepsilon) \leq S_1(s_0)$ for sufficiently small $\varepsilon \in (0, \varepsilon_0)$, which contradicts that $S'_1(s) > 0$ for each $s \in (s_0, s_0 + \varepsilon_0)$. Thus $(S_1(s), S_2(s))$ is not exceptional for each $s \in (0, s_*)$ due to (2.41), (2.35) and Proposition 2.2. If $s_* = 1$, then either $S_- = S$, or $S_- = \{(S_1(s), S_2(s)) : 0 \leq s < 1\}$ and $S_e = \{(S_1(1), S_2(1))\}$. Otherwise, $0 < s_* < 1$ and $S'_1(s_*) = 0$. Set

$$s^* = \sup\{s_* \leq s \leq 1 : S'_1(s) = 0\}.$$

Then, $0 < s_* \leq s^* \leq 1$. Similar to the discussion in (i), one can get that

$$S'_1(s) = 0, \quad s_* \leq s \leq s^*;$$

furthermore, $S_- = \{(S_1(s), S_2(s)) : 0 \leq s < s_*\}$ and $S_e = \{(S_1(1), S_2(1)) : s_* \leq s \leq 1\}$ if $s^* = 1$, while $S_- = \{(S_1(s), S_2(s)) : 0 \leq s < s_*\}$, $S_e = \{(S_1(1), S_2(1)) : s_* \leq s \leq s^*\}$ and $S_+ = \{(S_1(s), S_2(s)) : s^* < s \leq 1\}$ if $s^* < 1$.

(iv) The case that $(S_1(0), S_2(0))$ is not exceptional and $S'_1(0) < 0$. As the proof of (ii), one can prove that $S'_1(s) < 0$ and $(S_1(s), S_2(s))$ is not exceptional for each $s \in [0, 1]$. Then $S_+ = S$. \square

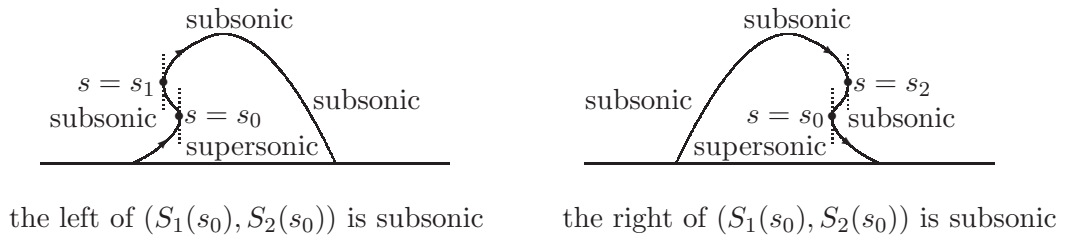
Theorem 2.2 *For any $C^2(\overline{G})$ transonic flow of Taylor type, each sonic point is not exceptional and φ is strictly monotone along the sonic curve.*

Proof. Without loss of generality, we assume that S is the sonic curve intersecting the lower wall. Denote

$$S : \varphi = S_1(s), \psi = S_2(s), (S'_1(s))^2 + (S'_2(s))^2 > 0, \quad 0 \leq s \leq 1, \quad S_1, S_2 \in C^2([0, 1])$$

with $S_2(0) = S_2(1) = \psi_2$ and $S_1(0) < S_1(1)$. Lemma 2.6 implies that $S'_1(0) \geq 0$ and $S'_1(1) \leq 0$.

First we show that there is no exceptional point in the nozzle. Otherwise, it is assumed that $(S_1(s_0), S_2(s_0))$ is an exceptional point for some $s_0 \in (0, 1)$. Owing to Remark 2.4, $S'_2(s_0) > 0$ and $S_1(s_0)$ achieves a local maximum if the left of $(S_1(s_0), S_2(s_0))$ is subsonic, while $S'_2(s_0) < 0$ and $S_1(s_0)$ achieves a local minimum if the right of $(S_1(s_0), S_2(s_0))$ is subsonic. Note that $S_2(0) = S_2(1)$ and $S_1(0) < S_1(1)$. If the left of $(S_1(s_0), S_2(s_0))$ is subsonic, then Lemma 2.7 yields that there exists $s_1 \in (s_0, 1)$ such that $S'_2(s_1) > 0$, $S_1(s_1)$ achieves a local strict minimum and the left of $(S_1(s_1), S_2(s_1))$ is subsonic. Similarly, if the right of $(S_1(s_0), S_2(s_0))$ is subsonic, then there exists $s_2 \in (0, s_0)$ such that $S'_2(s_2) < 0$, $S_1(s_2)$ achieves a local strict maximum and the right of $(S_1(s_2), S_2(s_2))$ is subsonic. Each case contradicts Lemma 2.6. Therefore, $(S_1(s), S_2(s))$ is not exceptional for each $s \in (0, 1)$. Furthermore, Proposition 2.2 implies that S_1 is a strictly increasing function on $[0, 1]$.



Let us show that $(S_1(0), S_2(0))$ is not exceptional by contradiction. Otherwise, $S'_1(0) \geq 0$ and Lemma 2.5 yield that S near $(S_1(0), S_2(0))$ is a graph of function with respect to ψ . Then, Propositions 2.2, 2.3 imply that $S_1(s) \leq S_1(0)$ for sufficiently small $s > 0$, which contradicts that S_1 is strictly increasing on $[0, 1]$. Similarly, one can prove that $(S_1(1), S_2(1))$ is not exceptional either. \square

Theorems 2.1 and 2.2 can be described in the physical plane. Assume that the nozzle in the physical plane is of the following general form

$$\mathcal{G} = \{(x, y) : f_1(x) < y < f_2(x), l_1 < x < l_2\},$$

where $l_1 < l_2$ and

$$f_1(x) < f_2(x), \quad l_1 \leq x \leq l_2.$$

The counter part of Theorem 2.1 is the following

Theorem 2.3 *For any C^2 transonic flow of Meyer type in the nozzle \mathcal{G} , its sonic curve \mathcal{S} is a disjoint union of three connected parts (may be empty): \mathcal{S}_e , \mathcal{S}_+ , \mathcal{S}_- , where \mathcal{S}_e is the set of exceptional points, which is a closed line segment, while \mathcal{S}_+ and \mathcal{S}_- denote the other two connected parts approaching the upper and lower walls, respectively. Moreover, if \mathcal{S}_e is empty, then $\mathcal{S} = \mathcal{S}_+$ or $\mathcal{S} = \mathcal{S}_-$.*

Similarly, in the physical plan, Theorem 2.2 becomes

Theorem 2.4 *For any C^2 transonic flow of Taylor type in the nozzle \mathcal{G} , each point on its sonic curve is not exceptional.*

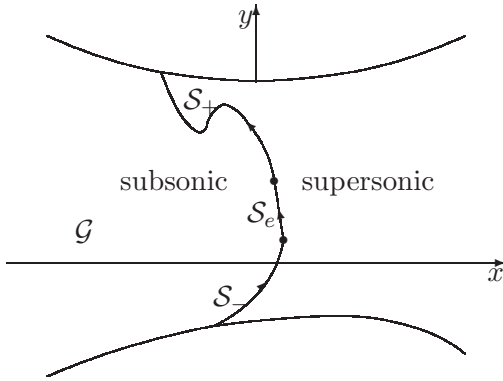


Figure of Theorem 2.3

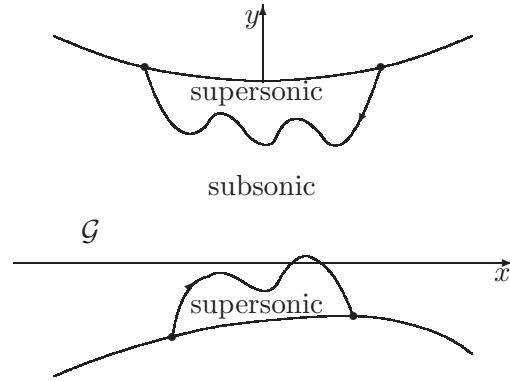


Figure of Theorem 2.4

2.6 Characteristics from sonic points

Characteristics play an important role in understanding the behavior of supersonic flows near sonic curves. So we first study characteristics from sonic points.

As will be shown in § 3.3, in the supersonic region, it holds that

$$\frac{\partial W}{\partial \psi}(\varphi, \psi) + \beta(\varphi, \psi) \frac{\partial W}{\partial \varphi}(\varphi, \psi) + \omega(\varphi, \psi) W(\varphi, \psi) = 0, \quad (2.42)$$

$$\frac{\partial Z}{\partial \psi}(\varphi, \psi) - \beta(\varphi, \psi) \frac{\partial Z}{\partial \varphi}(\varphi, \psi) + \omega(\varphi, \psi) Z(\varphi, \psi) = 0, \quad (2.43)$$

where

$$\beta(\varphi, \psi) = \left(\frac{-A'(q(\varphi, \psi))}{B'(q(\varphi, \psi))} \right)^{1/2} = \left(\frac{\gamma+1}{2} q^2(\varphi, \psi) - 1 \right)^{1/2} \left(1 - \frac{\gamma-1}{2} q^2(\varphi, \psi) \right)^{-1/(\gamma-1)-1/2},$$

$$\omega(\varphi, \psi) = \frac{\gamma+1}{2} q^4(\varphi, \psi) \left(\frac{\gamma+1}{2} q^2(\varphi, \psi) - 1 \right)^{-2} \left(1 - \frac{\gamma-1}{2} q^2(\varphi, \psi) \right)^{1/(\gamma-1)} A'(q(\varphi, \psi)) \frac{\partial q}{\partial \psi}(\varphi, \psi),$$

$$W(\varphi, \psi) = A'(q(\varphi, \psi)) \frac{\partial q}{\partial \varphi}(\varphi, \psi) - \frac{A'(q(\varphi, \psi))}{\beta(\varphi, \psi)} \frac{\partial q}{\partial \psi}(\varphi, \psi),$$

$$Z(\varphi, \psi) = -A'(q(\varphi, \psi)) \frac{\partial q}{\partial \varphi}(\varphi, \psi) - \frac{A'(q(\varphi, \psi))}{\beta(\varphi, \psi)} \frac{\partial q}{\partial \psi}(\varphi, \psi).$$

Note that β is positive in the supersonic region and vanishes at the sonic curve. The positive and negative characteristics of the system (2.42), (2.43) are governed by

$$\Phi'_+(\psi) = \beta(\Phi'_+(\psi), \psi) \quad \text{and} \quad \Phi'_-(\psi) = -\beta(\Phi'_-(\psi), \psi),$$

respectively.

The following a priori estimates hold along characteristics.

Lemma 2.8 *Assume that Σ_+ (Σ_-) is a positive (negative) characteristic. Then, $\theta(\Phi_\pm(\psi), \psi) \mp H(q(\Phi_\pm(\psi), \psi))$ is invariant on Σ_\pm , where*

$$H(q) = \int_{c_*}^q (-A'(s)B'(s))^{1/2} ds, \quad c_* \leq q < q_{max}.$$

Proof. On Σ_\pm , it holds that

$$\begin{aligned} \frac{d}{d\psi} \theta(\Phi_\pm(\psi), \psi) &= \pm \beta(\Phi_\pm(\psi), \psi) \frac{\partial \theta}{\partial \varphi}(\Phi_\pm(\psi), \psi) + \frac{\partial \theta}{\partial \psi}(\Phi_\pm(\psi), \psi) \\ &= \pm \left(-A'(q(\Phi_\pm(\psi), \psi)) B'(q(\Phi_\pm(\psi), \psi)) \right)^{1/2} \frac{\partial q}{\partial \psi}(\Phi_\pm(\psi), \psi) \\ &\quad - A'(q(\Phi_\pm(\psi), \psi)) \frac{\partial q}{\partial \varphi}(\Phi_\pm(\psi), \psi) \\ &= \pm \left(\frac{\partial H(q)}{\partial \psi}(\Phi_\pm(\psi), \psi) \pm \beta(\Phi_\pm(\psi), \psi) \frac{\partial H(q)}{\partial \varphi}(\Phi_\pm(\psi), \psi) \right) \\ &= \pm \frac{d}{d\psi} H(q(\Phi_\pm(\psi), \psi)). \end{aligned}$$

So, $\theta(\Phi_\pm(\psi), \psi) \mp H(q(\Phi_\pm(\psi), \psi))$ is invariant on Σ_\pm . \square

Consider a C^2 transonic flow of Meyer type in \mathcal{G} whose sonic curve \mathcal{S} intersects the upper wall at $(x_1, f_1(x_1))$. Assume that the subsonic region is located on the left of \mathcal{S} , $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_e$, \mathcal{S}_+ is not empty and \mathcal{S}_e is a line segment with nonempty interior. In the potential plane, the lower wall corresponds to $\psi = 0$ and the sonic curve is given by

$$S : \varphi = S_1(s), \psi = S_2(s), (S'_1(s))^2 + (S'_2(s))^2 > 0, \quad 0 \leq s \leq 1, \quad S_1, S_2 \in C^2([0, 1])$$

with $S_2(0) = 0$ and $S_2(1) = m$. Set

$$s^* = \sup \{s : S'_1(s) = 0, 0 \leq s \leq 1\}.$$

Then, as proved in Theorem 2.1, $0 < s^* < 1$,

$$S'_1(s) = 0, \quad \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) = 0, \quad 0 \leq s \leq s^*, \quad (2.44)$$

$$S'_1(s) < 0, \quad \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) > 0, \quad s^* < s \leq 1. \quad (2.45)$$

Let us first study characteristics from $(S_1(s), S_2(s))$ for $0 \leq s < s^*$.

Lemma 2.9 *Assume that $0 \leq s < s^*$. Then the positive and negative characteristics from $(S_1(s), S_2(s))$ coincide and are given by*

$$\Phi_+(\psi) = \Phi_-(\psi) = S_1(s), \quad 0 \leq \psi \leq S_2(s^*). \quad (2.46)$$

Proof. It is not hard to check that the functions given by (2.46) are positive and negative characteristics from $(S_1(s), S_2(s))$, respectively. Let us verify the uniqueness of the positive characteristic. Assume that $\tilde{\Phi}$ is a positive characteristic from $(S_1(s), S_2(s))$. Since $\tilde{\Phi}$ is a nondecreasing function, it suffices to verify that $\tilde{\Phi}$ coincide with the functions given by (2.46) in a right neighborhood of $S_2(s)$. Owing to (2.3) and (2.44), one gets that

$$A''(c_*) \left(\frac{\partial q}{\partial \varphi}(S_1(s), \psi) \right)^2 + B'(c_*) \frac{\partial^2 q}{\partial \psi^2}(S_1(s), \psi) + B''(c_*) \left(\frac{\partial q}{\partial \psi}(S_1(s), \psi) \right)^2 = 0, \quad 0 \leq \psi \leq S_2(s^*)$$

and

$$\frac{\partial q}{\partial \psi}(S_1(s), \psi) = 0, \quad \frac{\partial^2 q}{\partial \psi^2}(S_1(s), \psi) = 0, \quad 0 \leq \psi \leq S_2(s^*),$$

which imply

$$\frac{\partial q}{\partial \varphi}(S_1(s), \psi) = 0, \quad 0 \leq \psi \leq S_2(s^*). \quad (2.47)$$

Fix a small positive number δ such that $(S_1(s), S_1(s) + \delta] \times [S_2(s), S_2(s) + \delta]$ belongs to the supersonic region. Due to (2.47), there exists $M > 0$ such that

$$0 \leq \beta(\varphi, \psi) \leq M(\varphi - S_1(s)), \quad S_1(s) \leq \varphi \leq S_1(s) + \delta, \quad S_2(s) \leq \psi \leq S_2(s) + \delta. \quad (2.48)$$

Choose a number $\varepsilon \in (0, \delta)$ such that $\tilde{\Phi}(S_2(s) + \varepsilon) \leq S_1(s) + \delta$. Owing to (2.48), $\tilde{\Phi}$ satisfies

$$0 \leq \tilde{\Phi}'(\psi) \leq M(\tilde{\Phi}(\psi) - S_1(s)), \quad S_2(s) \leq \psi \leq S_2(s) + \varepsilon; \quad \tilde{\Phi}(S_2(s)) = S_1(s). \quad (2.49)$$

The unique solution to (2.49) is

$$\tilde{\Phi}(\psi) = S_1(s), \quad S_2(s) \leq \psi \leq S_2(s) + \varepsilon.$$

The uniqueness of the negative characteristic can be proved similarly. □

We now turn to characteristics from $(S_1(s), S_2(s))$ for $s^* < s < 1$.

Lemma 2.10 *Assume that $s^* < s < 1$. Then, there exist uniquely a positive and a negative characteristics from $(S_1(s), S_2(s))$, which contain no other sonic point except $(S_1(s), S_2(s))$.*

Proof. According to (2.45), the sonic curve from $(S_1(s^*), S_2(s^*))$ to $(S_1(1), S_2(1))$ is a graph of function with respect to φ , which is denoted by

$$\psi = S(\varphi), \quad S_1(1) \leq \varphi \leq S_1(s^*)$$

with

$$S'(\varphi) = -\frac{\partial q}{\partial \varphi}(\varphi, S(\varphi)) \left(\frac{\partial q}{\partial \psi}(\varphi, S(\varphi)) \right)^{-1} \in (-\infty, +\infty), \quad S_1(1) \leq \varphi < S_1(s^*). \quad (2.50)$$

First we show that $(S_1(s), S_2(s))$ is the unique intersecting point between any characteristic from $(S_1(s), S_2(s))$ and the sonic curve. Assume that Φ is a characteristic from $(S_1(s), S_2(s))$, which is defined in a right neighborhood of $S_2(s)$. Owing to (2.50) and $\Phi'(S_2(s)) = 0$, the path of Φ leaves for the supersonic region after $\psi = S_2(s)$ locally. Therefore, Φ is strictly monotone near $\psi = S_2(s)$ and its inverse function is denoted by Ψ , which solves

$$\Psi'(\varphi) = \frac{1}{\beta(\varphi, \Psi(\varphi))}, \quad \varphi > S_1(s), \quad \Psi(S_1(s)) = S_2(s)$$

if Φ is a positive characteristic, while

$$\Psi'(\varphi) = -\frac{1}{\beta(\varphi, \Psi(\varphi))}, \quad \varphi < S_1(s), \quad \Psi(S_1(s)) = S_2(s)$$

if Φ is a negative characteristic. Since $\Psi' = \infty$ at sonic points, it follows from (2.50) that the path of Ψ never approaches $\{(S_1(\tilde{s}), S_2(\tilde{s})) : s^* < \tilde{s} \leq 1\}$ after it leaves $(S_1(s), S_2(s))$. Let $s_0 \in [0, (s + s^*)/2]$ satisfy

$$S_2(s_0) = \max \left\{ S_2(\tilde{s}) : 0 \leq \tilde{s} \leq \frac{s + s^*}{2} \right\}.$$

Then, $s_0 > s^*$ since $\mathcal{S} \in C^2$ and \mathcal{S}_e is a segment parallel to the ψ -axis. Owing to the monotonicity of Ψ , the path of Ψ never approaches $\{(S_1(\tilde{s}), S_2(\tilde{s})) : 0 \leq \tilde{s} \leq s_0\}$ if Φ is a positive characteristic, while never approaches $\{(S_1(\tilde{s}), S_2(\tilde{s})) : 0 \leq \tilde{s} < s\}$ if Φ is a negative characteristic. Summing up, $(S_1(s), S_2(s))$ is the unique intersecting point between the path of Ψ and the sonic curve.

Below we prove the local existence and the uniqueness of negative characteristics under the additional assumption $S'(S_1(s)) < 0$, and the other cases can be discussed similarly. Since $q(S_1(s), S_2(s)) = c_*$, there exists a positive number τ_0 such that

$$G_0 = \{(\varphi, \psi) : S_1(s) - \tau_0 < \varphi < S_1(s), S(\varphi) < \psi < S(\varphi) + \tau_0\}$$

belongs to the supersonic region and

$$\frac{1}{\beta(\varphi, \psi)} \geq 1 - \min_{[S_1(s) - \tau_0, S_1(s)]} S', \quad (\varphi, \psi) \in G_0. \quad (2.51)$$

For $0 < \varepsilon \leq \tau_0/2$, $q(S_1(s), S_2(s) + \varepsilon) > c_*$. Therefore, there exists uniquely a solution Ψ_ε to the following problem

$$\Psi'_\varepsilon(\varphi) = -\frac{1}{\beta(\varphi, \Psi_\varepsilon(\varphi))}, \quad \varphi < S_1(s); \quad \Psi_\varepsilon(S_1(s)) = S_2(s) + \varepsilon. \quad (2.52)$$

Let $[\varphi_\varepsilon, S_1(s)]$ be the maximum interval of existence for the solution to the problem (2.52) on $\overline{G_0}$. It follows from (2.51) that

$$S(\varphi) + (S_1(s) - \varphi) \leq S_2(s) + \varepsilon + \left(1 - \min_{[S_1(s) - \tau_0, S_1(s)]} S'\right)(S_1(s) - \varphi) \leq \Psi_\varepsilon(\varphi), \quad \varphi_\varepsilon \leq \varphi \leq S_1(s), \quad (2.53)$$

which shows that the path of Ψ_ε does not intersect $\partial G_0 \cap S$. Therefore, for $0 < \varepsilon_1 < \varepsilon_2 \leq \tau_0/2$,

$$\varphi_{\varepsilon_1} \leq \varphi_{\varepsilon_2} \leq \varphi_{\tau_0/2} \quad \text{and} \quad \Psi_{\varepsilon_1}(\varphi) < \Psi_{\varepsilon_2}(\varphi) \quad \text{for each } \varphi_{\tau_0/2} \leq \varphi \leq S_1(s). \quad (2.54)$$

Let

$$\Psi(\varphi) = \lim_{\varepsilon \rightarrow 0^+} \Psi_\varepsilon(\varphi), \quad \varphi_{\tau_0/2} \leq \varphi \leq S_1(s).$$

Then, it follows from (2.52)–(2.54) that Ψ solves the following problem

$$\Psi'(\varphi) = -\frac{1}{\beta(\varphi, \Psi(\varphi))}, \quad \varphi_{\tau_0/2} < \varphi < S_1(s); \quad \Psi(S_1(s)) = S_2(s) \quad (2.55)$$

and satisfies

$$S(\varphi) + (S_1(s) - \varphi) \leq \Psi(\varphi) < \Psi_{\tau_0/2}(\varphi), \quad \varphi_{\tau_0/2} \leq \varphi \leq S_1(s). \quad (2.56)$$

Note that (2.55) and (2.56) imply

$$\Psi'(\varphi) < 0, \quad \varphi_{\tau_0/2} \leq \varphi < S_1(s).$$

Thus Ψ is strictly decreasing and its inverse function is a negative characteristic from $(S_1(s), S_2(s))$ due to (2.55).

Let us turn to the uniqueness. Assume that Φ and $\tilde{\Phi}$ are two negative characteristics from $(S_1(s), S_2(s))$, which are defined in a right neighborhood of $S_2(s)$. Since $\frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) > 0$ from (2.45), there exists a positive number $\tau_1 (\leq \tau_0)$ such that

$$\frac{1}{2} \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) \leq \frac{\partial q}{\partial \psi}(\varphi, \psi) \leq \frac{3}{2} \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)), \quad (\varphi, \psi) \in G_1, \quad (2.57)$$

where

$$G_1 = \{(\varphi, \psi) : S_1(s) - \tau_1 < \varphi < S_1(s), S(\varphi) < \psi < S(\varphi) + \tau_1\}.$$

Since $\Phi'(S_2(s)) = \tilde{\Phi}'(S_2(s)) = 0$, there exists a positive number δ so small that

$$(\Phi(\psi), \psi), (\tilde{\Phi}(\psi), \psi) \in G_1, \quad S_2(s) < \psi < S_2(s) + \delta \quad (2.58)$$

and

$$0 \leq \sup_{G_1} \left| \frac{\partial q}{\partial \varphi} \right| (S_1(s) - \Phi(\psi)) \leq \frac{1}{4} \frac{\partial q}{\partial \psi}(S_1(s), S_2(s))(\psi - S_2(s)), \quad S_2(s) \leq \psi \leq S_2(s) + \delta, \quad (2.59)$$

$$0 \leq \sup_{G_1} \left| \frac{\partial q}{\partial \varphi} \right| (S_1(s) - \tilde{\Phi}(\psi)) \leq \frac{1}{4} \frac{\partial q}{\partial \psi}(S_1(s), S_2(s))(\psi - S_2(s)), \quad S_2(s) \leq \psi \leq S_2(s) + \delta. \quad (2.60)$$

Set

$$h(\psi) = (\Phi(\psi) - \tilde{\Phi}(\psi))^2, \quad S_2(s) \leq \psi \leq S_2(s) + \delta.$$

Then, h solves

$$h'(\psi) = \xi(\psi)(q(\zeta(\psi), \psi) - c_*)^{-1/2} h(\psi), \quad S_2(s) < \psi < S_2(s) + \delta; \quad h(S_2(s)) = 0, \quad (2.61)$$

where $\xi \in C([S_2(s), S_2(s) + \delta])$, while

$$\min\{\Phi(\psi), \tilde{\Phi}(\psi)\} \leq \zeta(\psi) \leq \max\{\Phi(\psi), \tilde{\Phi}(\psi)\}, \quad S_2(s) < \psi < S_2(s) + \delta. \quad (2.62)$$

It follows from (2.57)–(2.60) and (2.62) that

$$\begin{aligned} q(\zeta(\psi), \psi) - c_* &\leq \sup_{G_1} \left| \frac{\partial q}{\partial \varphi} \right| (S_1(s) - \zeta(\psi)) + \frac{3}{2} \frac{\partial q}{\partial \psi} (S_1(s), S_2(s)) (\psi - S_2(s)) \\ &\leq \frac{7}{4} \frac{\partial q}{\partial \psi} (S_1(s), S_2(s)) (\psi - S_2(s)), \quad S_2(s) < \psi < S_2(s) + \delta, \end{aligned} \quad (2.63)$$

$$\begin{aligned} q(\zeta(\psi), \psi) - c_* &\geq - \sup_{G_1} \left| \frac{\partial q}{\partial \varphi} \right| (S_1(s) - \zeta(\psi)) + \frac{1}{2} \frac{\partial q}{\partial \psi} (S_1(s), S_2(s)) (\psi - S_2(s)) \\ &\geq \frac{1}{4} \frac{\partial q}{\partial \psi} (S_1(s), S_2(s)) (\psi - S_2(s)), \quad S_2(s) < \psi < S_2(s) + \delta. \end{aligned} \quad (2.64)$$

Substituting (2.63) and (2.64) into (2.61) yields

$$|h'(\psi)| \leq M(\psi - S_2(s))^{-1/2} h(\psi), \quad S_2(s) < \psi < S_2(s) + \delta; \quad h(S_2(s)) = 0 \quad (2.65)$$

for some positive constant M . Since (2.65) admits only a trivial solution, the uniqueness is proved. \square

For fixed $s \in (s^*, 1)$, according to Lemma 2.10, there exist uniquely a positive and a negative characteristics from $(S_1(s), S_2(s))$; furthermore, for each characteristic, the lower endpoint is $(S_1(s), S_2(s))$, the upper endpoint is located at the upper wall or the outlet of the nozzle. Without loss of generality, the nozzle is assumed to be so long that all upper endpoints of the characteristics from $(S_1(s), S_2(s))$ for $s \in (s^*, 1)$ are located at the upper wall. Therefore, the positive and the negative characteristics from $(S_1(s), S_2(s))$ are governed by

$$\Sigma_{s,+} : \varphi = \Phi_{s,+}(\psi) : \begin{cases} \frac{d}{d\psi} \Phi_{s,+}(\psi) = \beta(\Psi_{s,+}(\psi), \psi), & S_2(s) < \psi < m, \\ \Phi_{s,+}(S_2(s)) = S_1(s), \quad \Phi_{s,+}(m) = \varphi_{s,+} \end{cases}$$

and

$$\Sigma_{s,-} : \varphi = \Phi_{s,-}(\psi) : \begin{cases} \frac{d}{d\psi} \Phi_{s,-}(\psi) = -\beta(\Psi_{s,-}(\psi), \psi), & S_2(s) < \psi < m, \\ \Phi_{s,-}(S_2(s)) = S_1(s), \quad \Phi_{s,-}(m) = \varphi_{s,-}, \end{cases}$$

respectively, where $s \in (s^*, 1)$.

For $s^* < s_1 < s_2 < 1$, the uniqueness of the characteristic shows

$$\varphi_{s_1,+} > \varphi_{s_2,+}, \quad \varphi_{s_1,-} > \varphi_{s_2,-}$$

and

$$\Phi_{s_1,+}(\psi) > \Phi_{s_2,+}(\psi), \quad \Phi_{s_1,-}(\psi) > \Phi_{s_2,-}(\psi), \quad S_2(s_2) \leq \psi \leq m.$$

Set

$$\varphi_- = \lim_{s \rightarrow (s^*)^+} \varphi_{s,-}, \quad \varphi_+ = \lim_{s \rightarrow (s^*)^+} \varphi_{s,+}. \quad (2.66)$$

Remark 2.7 *It is not clear at this moment whether the positive and negative characteristics from $(S_1(s^*), S_2(s^*))$ are unique. However,*

$$\left\{ \begin{array}{ll} S_1(s^*), & 0 \leq \psi \leq S_2(s^*), \\ \lim_{s \rightarrow (s^*)^+} \Phi_{s,+}(\psi), & S_2(s^*) < \psi \leq m \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} S_1(s^*), & 0 \leq \psi \leq S_2(s^*), \\ \lim_{s \rightarrow (s^*)^+} \Phi_{s,-}(\psi), & S_2(s^*) < \psi \leq m \end{array} \right.$$

are the maximal positive and the minimal negative characteristics from $(S_1(s^), S_2(s^*))$, respectively.*

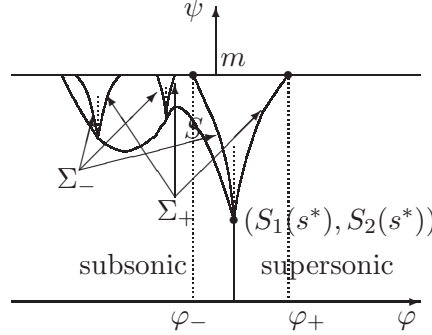


Figure: characteristics for the transonic flow in the potential plane

Let x^* be the point such that $(x^*, f_1(x^*))$ is transformed to (φ_-, m) in the coordinates transformation from the physical plane to the potential plane. That is to say,

$$x^* = \sup \{x \in [l_1, l_2] : (x, f_1(x)) \text{ is the endpoint of the negative characteristic from a point on } \mathcal{S}_+\}. \quad (2.67)$$

Proposition 2.4 *For each $x \in [x_1, x^*]$, $f_1''(x) > 0$.*

Proof. Fix $s \in (s^*, 1)$. It follows from the definition of W that

$$\frac{W(\varphi, \Psi_{s,+}(\varphi))}{A'(q(\varphi, \Psi_{s,+}(\varphi)))} = \frac{\partial q}{\partial \varphi}(\varphi, \Psi_{s,+}(\varphi)) - \frac{1}{\beta(\varphi, \Psi_{s,+}(\varphi))} \frac{\partial q}{\partial \psi}(\varphi, \Psi_{s,+}(\varphi)), \quad S_1(s) < \varphi \leq \varphi_{s,+}.$$

Since $\beta(S_1(s), S_2(s)) = 0$ and $\frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) > 0$, there exists a positive number $\delta (\leq \varphi_{s,+} - S_1(s))$ such that $W(S_1(s) + \delta, \Psi_{s,+}(S_1(s) + \delta)) > 0$. Then, (2.42) yields

$$W(\varphi_{s,+}, m) > 0. \quad (2.68)$$

Similarly, one can get

$$Z(\varphi_{s,-}, m) > 0. \quad (2.69)$$

Due to the arbitrariness of $s \in (s^*, 1)$, it follows from (2.68) and (2.69) that

$$W(\varphi, m) > 0, \quad S_1(1) < \varphi < \varphi_+, \quad W(\varphi_+, m) \geq 0$$

and

$$Z(\varphi, m) > 0, \quad S_1(1) < \varphi < \varphi_-, \quad Z(\varphi_-, m) \geq 0.$$

Thus,

$$\frac{\partial q}{\partial \psi}(\varphi, m) = -\frac{\beta(\varphi, m)}{2A'(q(\varphi, m))}(W(\varphi, m) + Z(\varphi, m)) > 0, \quad S_1(1) < \varphi \leq \varphi_-. \quad (2.70)$$

Moreover, (2.45) gives

$$\frac{\partial q}{\partial \psi}(S_1(1), m) > 0. \quad (2.71)$$

Assume that $(x, f_1(x))$ is transformed into $(\Phi(x), m)$ for $l_1 < x < l_2$ in the coordinates transformation from the physical plane to the potential plane. Then,

$$\Phi'(x) = \frac{q(x, f_1(x))}{\cos \theta(x, f_1(x))} = q(x, f_1(x))\sqrt{1 + (f_1'(x))^2} > 0, \quad l_1 < x < l_2 \quad (2.72)$$

and

$$\Phi'(x)B'(q(\Phi(x), m))\frac{\partial q}{\partial \psi}(\Phi(x), m) = \frac{d\theta(x, f_1(x))}{dx} = \frac{f_1''(x)}{1 + (f_1'(x))^2}, \quad l_1 < x < l_2. \quad (2.73)$$

The conclusion of the proposition follows from (2.70)–(2.73). \square

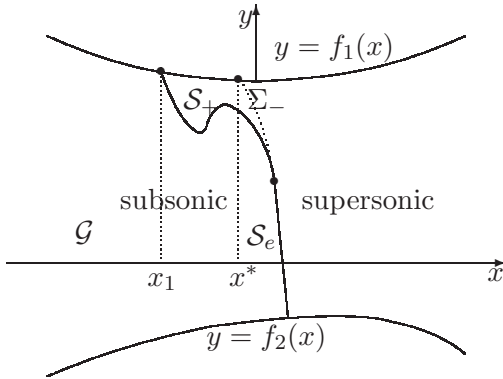


Figure of Proposition 2.4

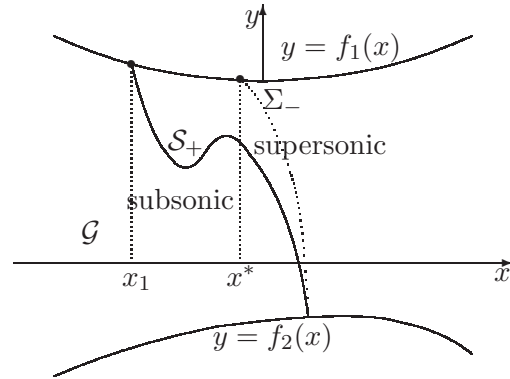


Figure of Remark 2.8

Remark 2.8 *Proposition 2.4 still holds if S_e is empty or a single point set.*

Similar to Lemmas 2.9, 2.10, Proposition 2.4 and Remarks 2.7, 2.8, one can prove the following two general conclusions. One is on characteristics from sonic points, and the other is on the geometry of walls.

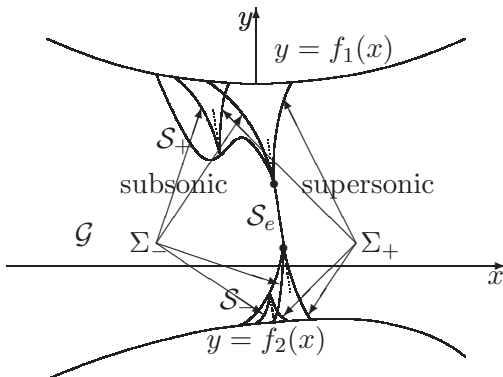


Figure of Theorem 2.5

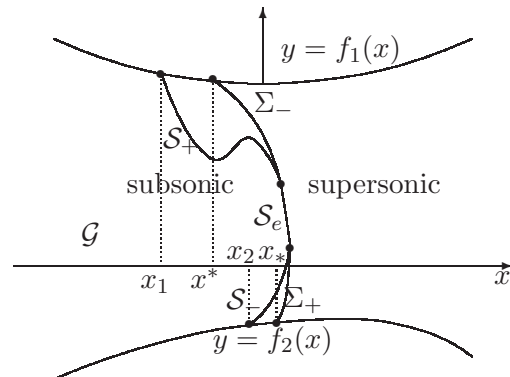


Figure of Theorem 2.6

Theorem 2.5 Let $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_e \cup \mathcal{S}_-$ be the sonic curve of a C^2 transonic flow of Meyer type in \mathcal{G} .

- (i) There exist uniquely a positive and a negative characteristics from each interior point of $\mathcal{S}_+ \cup \mathcal{S}_-$.
- (ii) All positive and negative characteristics from points of $\mathcal{S}_e \setminus \partial(\mathcal{S}_+ \cup \mathcal{S}_-)$ are the same line \mathcal{S}_e .
- (iii) There are positive and negative characteristics from points of $\mathcal{S}_e \cap \partial(\mathcal{S}_+ \cup \mathcal{S}_-)$. Particularly, the maximal positive and the minimal negative characteristics are unique.

Characteristics from boundary sonic points are as follows.

Remark 2.9 Let $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_e \cup \mathcal{S}_-$ be the sonic curve of a C^2 transonic flow of Meyer type in \mathcal{G} . Assume that the subsonic region is located on the left. For the sonic point on the upper wall,

- (i) If it belongs to \mathcal{S}_+ , then there is not any characteristic from this point.
- (ii) If it belongs to $\mathcal{S}_e \setminus \partial\mathcal{S}_-$, then the positive and negative characteristics from this point are the same line \mathcal{S}_e .
- (iii) If it belongs to $\mathcal{S}_e \cap \partial\mathcal{S}_-$, then there are positive and negative characteristics from this point. Particularly, the maximal positive and the minimal negative characteristics are unique.
- (iv) If it belongs to \mathcal{S}_- , then there exist uniquely a positive and a negative characteristics from this point.

For the sonic point on the lower wall,

- (i) If it belongs to \mathcal{S}_- , then there is not any characteristic from this point.
- (ii) If it belongs to $\mathcal{S}_e \setminus \partial\mathcal{S}_+$, then the positive and negative characteristics from this point are the same line \mathcal{S}_e .
- (iii) If it belongs to $\mathcal{S}_e \cap \partial\mathcal{S}_+$, then there are positive and negative characteristics from this point. Particularly, the maximal positive and the minimal negative characteristics are unique.
- (iv) If it belongs to \mathcal{S}_+ , then there exist uniquely a positive and a negative characteristics from this point.

Theorem 2.6 Let $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_e \cup \mathcal{S}_-$ be the sonic curve of a C^2 transonic flow of Meyer type in \mathcal{G} . Assume that the subsonic region is located on the left and \mathcal{S} intersects the upper and lower walls at $(x_1, f_1(x_1))$ and $(x_2, f_2(x_2))$, respectively.

- (i) If $\mathcal{S}_+ \neq \emptyset$, then $f_1''(x) > 0$ for each $x \in [x_1, x^*]$, where x^* is given by (2.67).
- (ii) If $\mathcal{S}_- \neq \emptyset$, then $f_2''(x) < 0$ for each $x \in [x_2, x_*]$, where

$$x_* = \sup \{x \in [l_1, l_2] : (x, f_2(x)) \text{ is the endpoint of the positive characteric from a point on } \mathcal{S}_-\}. \quad (2.74)$$

As a consequence of Theorem 2.4 and the proof of Proposition 2.4, we can obtain

Theorem 2.7 Consider a C^2 transonic flow of Taylor type in \mathcal{G} . Assume that the sonic curve intersects the upper wall at $(x_{1,-}, f_1(x_{1,-}))$ and $(x_{1,+}, f_1(x_{1,+}))$, while the lower wall at $(x_{2,-}, f_2(x_{2,-}))$ and $(x_{2,+}, f_2(x_{2,+}))$ with $x_{k,-} < x_{k,+}$ ($k = 1, 2$). Then,

$$f_1''(x) > 0, \quad x_{1,-} \leq x \leq x_{1,+}$$

and

$$f_2''(x) < 0, \quad x_{2,-} \leq x \leq x_{2,+}.$$

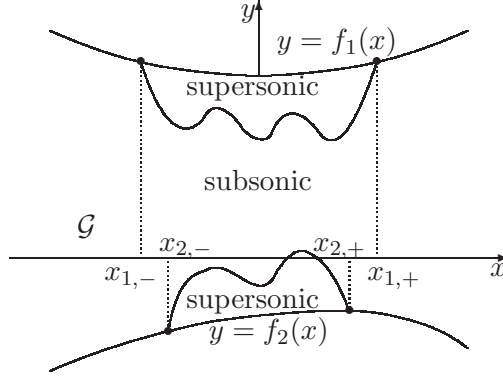


Figure of Theorem 2.7

Remark 2.10 For a smooth transonic flow of Taylor type, it has been mentioned in [4] that the boundary of a supersonic enclosure on a wall cannot contain a straight segment. Theorem 2.7 strengthens it to be that the curvature of this boundary must be nonzero.

2.7 Instability of transonic flows with nonexceptional points

According to Theorems 2.1 and 2.6, we are ready to show that C^2 transonic flows with nonexceptional points are unstable with respect to small changes in the shape of the nozzle.

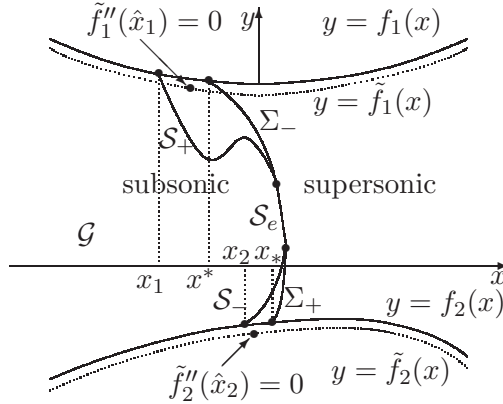


Figure of Theorem 2.8

Theorem 2.8 Consider a $C^2(\bar{\mathcal{G}})$ transonic flow of Meyer type, whose subsonic region is located on the left. Assume that the sonic curve $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_e \cup \mathcal{S}_-$ intersects the upper and lower walls at $(x_1, f_1(x_1))$ and $(x_2, f_2(x_2))$, respectively. If there is a nonexceptional point on \mathcal{S} , then φ is unstable with respect to small changes in the shape of the nozzle in the following sense.

(i) The case $\mathcal{S}_+ \neq \emptyset$. Let x^* be given by (2.67). If $\tilde{f}_1 \in C^\infty([l_1, l_2])$ is a C^1 small perturbation of f_1 and satisfies $\tilde{f}_1''(\hat{x}_1) = 0$ with some $x_1 < \hat{x}_1 < x^*$, then there is not a $C^2(\tilde{\mathcal{G}})$ transonic flow of Meyer type with $\tilde{\mathcal{G}}$ being the corresponding domain when the upper wall is replaced by \tilde{f}_1 , which is a C^0 small perturbation of the background flow.

(ii) The case $\mathcal{S}_- \neq \emptyset$. Let x_* be given by (2.74). If $\tilde{f}_2 \in C^\infty([l_1, l_2])$ is a C^1 small perturbation

of f_2 and satisfies $\tilde{f}_2''(\hat{x}_2) = 0$ with some $x_2 < \hat{x}_2 < x_*$, then there is not a $C^2(\tilde{\mathcal{G}})$ transonic flow of Meyer type with $\tilde{\mathcal{G}}$ being the corresponding domain when the lower wall is replaced by \tilde{f}_2 , which is a C^0 small perturbation of the background flow.

Proof. We prove (i) only and the proof of (ii) is similar. Consider a $C^2(\tilde{\mathcal{G}})$ transonic flow of Meyer type, whose sonic curve is denoted by $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_+ \cup \tilde{\mathcal{S}}_e \cup \tilde{\mathcal{S}}_-$ according to Theorem 2.1. If the transonic flow is close enough to the background flow in C^0 -norm, then $\tilde{\mathcal{S}}_+ \neq \emptyset$ and its two endpoints are small perturbations of the two endpoints of \mathcal{S}_+ , respectively. Thus, \tilde{x}_1 and \tilde{x}^* are small perturbations of x_1 and x^* , respectively, and it follows from Theorem 2.6 that

$$\tilde{f}_1''(x) > 0, \quad \tilde{x}_1 < x < \tilde{x}^*,$$

where \tilde{x}_1 and \tilde{x}^* are defined in a similar way as for x_1 and x^* , respectively. This contradicts that $\tilde{f}_1''(\tilde{x}_1) = 0$. \square

Remark 2.11 *The instability in Theorem 2.8 is weak. It is unknown whether the flow is unstable if the nozzle wall is perturbed in C^2 or other smooth spaces.*

Remark 2.12 *Theorem 2.8 motives us to seek a smooth transonic flow of Meyer type whose sonic points are exceptional. Indeed, we do prove the existence of such a smooth transonic flow for the symmetric de Laval nozzle Ω in the paper.*

Moreover, for C^2 transonic flows of Meyer type in Ω , one can prove

Proposition 2.5 *Assume that $f''(0) = 0$. Consider a C^2 transonic flow of Meyer type in Ω whose each streamline in the supersonic region is a graph of function with respect to x . If there is a nonexceptional point at the sonic curve, then not all streamlines are convex.*

Proof. It is assumed that the subsonic region is located on the left without loss of generality. We transform the flow from the physical plane to the potential plane. It is assumed that $(0, f(0))$ is transformed to $(0, m)$ and the sonic curve is transformed to

$$S : \varphi = S_1(s), \psi = S_2(s), (S_1'(s))^2 + (S_2'(s))^2 > 0, \quad 0 \leq s \leq 1$$

with $S_2(0) = 0$ and $S_2(1) = m$. Set

$$s^* = \sup \{s : S_1'(s) = 0, 0 \leq s \leq 1\}.$$

The symmetry of the nozzle and Proposition 2.2 show that $(S_1(0), S_2(0))$ is an exceptional point and $\theta(S_1(0), S_2(0)) = 0$. Similar to the discussion in §2.5, one can prove that $0 \leq s^* < 1$ and

$$\begin{aligned} S_1'(s) = 0, \quad \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) = 0, \quad \theta(S_1(s), S_2(s)) = 0, \quad 0 \leq s \leq s^*, \\ S_1'(s) < 0, \quad \frac{\partial q}{\partial \psi}(S_1(s), S_2(s)) > 0, \quad \theta(S_1(s), S_2(s)) < 0, \quad s^* < s \leq 1. \end{aligned}$$

Lemma 2.8 implies that

$$\theta(\varphi_-, m) + H(q(\varphi_-, m)) = 0, \quad \theta(\varphi_+, m) - H(q(\varphi_+, m)) = 0,$$

where φ_{\pm} are defined by (2.66) and H is given in Lemma 2.8. Thanks to $q(\varphi_{\pm}, m) > c_*$, one gets that $\theta(\varphi_-, m) < 0 < \theta(\varphi_+, m)$ and thus $\varphi_- < 0 < \varphi_+$. As shown in Proposition 2.4, one can get $W(0, m) > 0$ and thus $Z(0, m) = -W(0, m) < 0$. Denote Σ_- to be the negative characteristic from $(0, m)$. Due to Lemma 2.8, $\theta + H(q)$ is invariant on Σ_- , which equals identically to $\theta(0, m) + H(q(0, m)) > 0$. Note that

$$\theta(S_1(s), S_2(s)) + H(q(S_1(s), S_2(s))) = \theta(S_1(s), S_2(s)) \leq 0, \quad 0 \leq s \leq 1.$$

Thus Σ_- never approach the sonic curve. Assume that Σ_- intersects the lower wall at $(\varphi_1, 0)$. Then $S_1(0) < \varphi_1$. Since the flow is sonic at $(S_1(0), 0)$ and supersonic at $(\varphi_1, 0)$, there exists $\varphi_2 \in (S_1(0), \varphi_1)$ such that $W(\varphi_2, 0) < 0$. Let Σ_+ be the positive characteristic from $(\varphi_2, 0)$ and denote (φ_*, ψ_*) to be the intersecting point of Σ_+ and Σ_- . It follows from (2.42) and (2.43) that $W(\varphi_*, \psi_*) < 0$ and $Z(\varphi_*, \psi_*) < 0$, which imply

$$\frac{\partial q}{\partial \psi}(\varphi_*, \psi_*) = -\frac{\beta(\varphi_*, \psi_*)}{2A'(q(\varphi_*, \psi_*))}(W(\varphi_*, \psi_*) + Z(\varphi_*, \psi_*)) < 0.$$

Assume that $y = f_*(x)$ is the streamline across (φ_*, ψ_*) and $(x_*, f_*(x_*))$ corresponds to (φ_*, ψ_*) in the coordinates transformation. Then, as shown in (2.72) and (2.73), $f''_*(x_*)$ and $\frac{\partial q}{\partial \psi}(\varphi_*, \psi_*)$ are of the same sign. Thus $f''_*(x_*) < 0$. \square

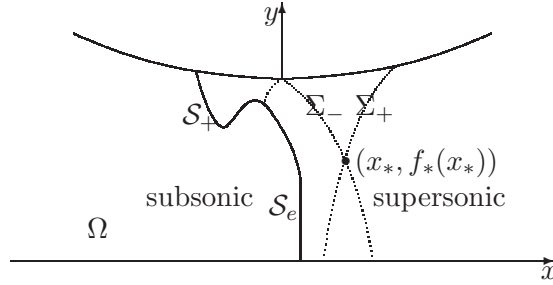


Figure of the proof of Proposition 2.5

3 Formulation of the smooth transonic flow problem and main results

As mentioned in Remark 2.12, we seek a smooth transonic flow of Meyer type, whose sonic points are exceptional, in the symmetric de Laval nozzle Ω . In this section, let us formulate this smooth transonic flow problem both in the physical plane and in the potential plane, and state the existence and uniqueness theorems.

3.1 Formulation of the transonic flow problem in the de Laval nozzle

To seek a smooth transonic flow in Ω whose sonic points are exceptional, the first step is to determine the location of the sonic curve. It is not hard to verify from Lemma 2.1 and Proposition 2.1 that

Proposition 3.1 *For a $C^2(\overline{\Omega})$ transonic flow of Meyer type, the following statements are equivalent*

- (i) *The endpoint of the sonic curve on the wall is located at the throat point;*
- (ii) *The flow angle always equals to zero at the sonic curve;*
- (iii) *Every point at the sonic curve is exceptional;*
- (iv) *The velocity vector is along the normal direction at the sonic curve;*
- (v) *The potential is a constant at the sonic curve;*
- (vi) *The sonic curve is located at the throat of the nozzle.*

Therefore, the sonic curve of the sought smooth transonic flow should be located at the throat. That is to say, the flow should be subsonic in the convergent part, then sonic at the throat and then supersonic in the divergent part. For such a smooth transonic flow, the velocity vector is along the normal direction at the sonic curve and thus the mass flux is

$$m = f(0)c_*^{1+2/(\gamma-1)}. \quad (3.1)$$

Furthermore, the potential of the flow is a constant at the sonic curve and we normalize it to be zero in the paper. Then, the potential of the flow at the inlet, which is a constant, is free. So is the potential at the outlet.

Thus, the problem of a transonic flow in Ω , whose velocity vector is along the normal direction at the inlet and the outlet, which satisfies the slip condition on the wall and whose sonic curve is located at the throat, can be formulated as

$$\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, \quad (x, y) \in \Omega, \quad (3.2)$$

$$\varphi(g(y), y) = C_{\text{in}}, \quad 0 < y < f(l_-), \quad (3.3)$$

$$\frac{\partial\varphi}{\partial y}(x, 0) = 0, \quad g(0) < x < t(0), \quad (3.4)$$

$$\frac{\partial\varphi}{\partial y}(x, f(x)) - f'(x)\frac{\partial\varphi}{\partial x}(x, f(x)) = 0, \quad l_- < x < l_+, \quad (3.5)$$

$$\varphi(t(y), y) = C_{\text{out}}, \quad 0 < y < f(l_+), \quad (3.6)$$

$$|\nabla\varphi(0, y)| = c_*, \quad \varphi(0, y) = 0, \quad 0 < y < f(0), \quad (3.7)$$

$$|\nabla\varphi(x, y)| < c_*, \quad (x, y) \in \Omega_-, \quad (3.8)$$

$$|\nabla\varphi(x, y)| > c_*, \quad (x, y) \in \Omega_+, \quad (3.9)$$

where $C_{\text{in}}, C_{\text{out}} \in \mathbb{R}$ are free, the outlet $\Gamma_{\text{out}} : x = t(y)$ ($0 \leq y \leq f(l_+)$) is free, Ω_- and Ω_+ are the convergent part and the divergent part of the nozzle, respectively, i.e.

$$\Omega_- = \{(x, y) \in \Omega : x < 0\}, \quad \Omega_+ = \{(x, y) \in \Omega : x > 0\}.$$

Since the sonic curve of the smooth transonic flow is located at the throat, we can decompose the transonic flow problem (3.2)–(3.9) into a subsonic-sonic flow problem and a sonic-supersonic flow problem as follows

$$\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, \quad (x, y) \in \Omega_-, \quad (3.10)$$

$$\varphi(g(y), y) = C_{\text{in}}, \quad 0 < y < f_-(l_-), \quad (3.11)$$

$$\frac{\partial\varphi}{\partial y}(x, 0) = 0, \quad g(0) < x < 0, \quad (3.12)$$

$$\frac{\partial \varphi}{\partial y}(x, f_-(x)) - f'_-(x) \frac{\partial \varphi}{\partial x}(x, f_-(x)) = 0, \quad l_- < x < 0, \quad (3.13)$$

$$|\nabla \varphi(0, y)| = c_*, \quad \varphi(0, y) = 0, \quad 0 < y < f_-(0), \quad (3.14)$$

$$|\nabla \varphi(x, y)| < c_*, \quad (x, y) \in \Omega_- \quad (3.15)$$

and

$$\operatorname{div}(\rho(|\nabla \varphi|^2) \nabla \varphi) = 0, \quad (x, y) \in \Omega_+, \quad (3.16)$$

$$|\nabla \varphi(0, y)| = c_*, \quad \varphi(0, y) = 0, \quad 0 < y < f_+(0), \quad (3.17)$$

$$\frac{\partial \varphi}{\partial y}(x, 0) = 0, \quad 0 < x < t(0), \quad (3.18)$$

$$\frac{\partial \varphi}{\partial y}(x, f_+(x)) - f'_+(x) \frac{\partial \varphi}{\partial x}(x, f_+(x)) = 0, \quad 0 < x < l_+, \quad (3.19)$$

$$\varphi(t(y), y) = C_{\text{out}}, \quad 0 < y < f_+(l_+), \quad (3.20)$$

$$|\nabla \varphi(x, y)| > c_*, \quad (x, y) \in \Omega_+, \quad (3.21)$$

where f_- and f_+ are the walls of the convergent part and divergent part of the nozzle, respectively, i.e.

$$f_-(x) = f(x), \quad l_- \leq x \leq 0 \quad \text{and} \quad f_+(x) = f(x), \quad 0 \leq x \leq l_+.$$

Remark 3.1 For the subsonic-sonic flow problem (3.10)–(3.15), there are two boundary conditions at the sonic curve (see (3.14)). Indeed, the potential at the sonic curve is a free constant, which is normalized to be zero and thus the potential at the inlet is free (see (3.11), where C_{in} is a free parameter).

It should be noted that to study a subsonic-sonic flow problem and a sonic-supersonic flow problem, one must determine and control precisely the speed of the flow near the sonic state, which plays a very essential role in the mathematical analysis. In (3.2), the speed of the flow is the absolute value of the gradient of a solution, which is very hard to estimate precisely. It turns out to be more convenient to solve the subsonic-sonic flow problem (3.10)–(3.15) and the sonic-supersonic flow problem (3.16)–(3.21) in the potential plane, where the speed of the flow is a solution to (2.3). We will thus formulate these two problems in the potential plane in next two subsections.

3.2 Formulation of the subsonic-sonic flow problem in the potential plane

We will show that there exists a subsonic-sonic flow in the divergent part of the nozzle Ω_- . In this subsection, we formulate this subsonic-sonic flow problem in the potential plane.

Assume that $f_- \in C^3([l_-, 0])$ ($-1 \leq l_- < 0$) satisfies

$$\delta_{1,-}(-x)^{\lambda_-} \leq f''_-(x) \leq \delta_{2,-}(-x)^{\lambda_-}, \quad l_- \leq x \leq 0 \quad (3.22)$$

with positive constants λ_- , $\delta_{1,-}$ and $\delta_{2,-}$ such that $\lambda_- > 2$ and $\delta_{1,-} \leq \delta_{2,-}$. Choose the inlet Γ_{in} as a small perturbation of the arc

$$\Gamma_0 : x = g_0(y), \quad 0 \leq y \leq f_-(l_-),$$

where

$$g_0(y) = l_- - \frac{f_-(l_-)}{f'_-(l_-)} - \sqrt{R_0^2 - y^2}, \quad 0 \leq y \leq f_-(l_-), \quad R_0 = \frac{f_-(l_-) \sqrt{(f'_-(l_-))^2 + 1}}{-f'_-(l_-)}.$$

More precisely,

$$\Gamma_{\text{in}} : x = g(y), \quad 0 \leq y \leq f_-(l_-)$$

with $g \in C^3([0, f_-(l_-)])$ satisfying

$$g'(0) = 0, \quad g(f_-(l_-)) = l_-, \quad g'(f_-(l_-)) = -f'_-(l_-) \quad (3.23)$$

and

$$\frac{1}{2R_0} \leq \left(\frac{g'(y)}{\sqrt{(g'(y))^2 + 1}} \right)' \leq \frac{3}{2R_0}, \quad \left| \left(\frac{g'(y)}{\sqrt{(g'(y))^2 + 1}} \right)'' \right| \leq \epsilon(l_-)(-l_-)^{3\lambda_-/2}, \quad 0 \leq y \leq f_-(l_-), \quad (3.24)$$

where $0 < \epsilon(s) \leq 1$ for $-1 \leq s < 0$ and $\lim_{s \rightarrow 0^-} \epsilon(s) = 0$.

Since the velocity vector of the flow is along the normal direction at the inlet and the wall of the nozzle is solid, the flow angle at the inlet and at the upper wall can be expressed as

$$\Theta_{\text{in}}(y) = -\arctan g'(y), \quad 0 \leq y \leq f_-(l_-)$$

and

$$\Theta_-(x) = \arctan f'_-(x), \quad l_- \leq x \leq 0,$$

respectively. Let the speed of the flow at the inlet and at the upper wall be denoted by

$$q(g(y), y) = \mathcal{Q}_{\text{in}}(y), \quad 0 \leq y \leq f_-(l_-)$$

and

$$q(x, f(x)) = \mathcal{Q}_-(x), \quad l_- \leq x \leq 0,$$

respectively. The incoming mass flux is given by

$$m_{\text{in}} = \int_0^{f_-(l_-)} \frac{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))}{\cos \Theta_{\text{in}}(y)} dy. \quad (3.25)$$

Since the potential at the sonic curve is normalized to be zero, the potential at the inlet is given by

$$\zeta_- = \int_0^{l_-} \frac{\mathcal{Q}_-(x)}{\cos \Theta_-(x)} dx. \quad (3.26)$$

At the inlet, the stream function is

$$\psi(g(y), y) = \Psi_{\text{in}}(y), \quad 0 \leq y \leq f_-(l_-)$$

satisfying $\Psi_{\text{in}}(0) = 0$, $\Psi_{\text{in}}(f_-(l_-)) = m_{\text{in}}$ and

$$\Psi'_{\text{in}}(y) = \frac{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))}{\cos \Theta_{\text{in}}(y)} > 0, \quad 0 \leq y \leq f_-(l_-).$$

Thus, Ψ_{in} is expressed as

$$\Psi_{\text{in}}(y) = \int_0^y \frac{\mathcal{Q}_{\text{in}}(s) \rho(\mathcal{Q}_{\text{in}}^2(s))}{\cos \Theta_{\text{in}}(s)} ds, \quad 0 \leq y \leq f_-(l_-). \quad (3.27)$$

Denote by Y_{in} the inverse function of Ψ_{in} , i.e.

$$Y_{\text{in}}(\psi) = \Psi_{\text{in}}^{-1}(\psi), \quad 0 \leq \psi \leq m_{\text{in}}. \quad (3.28)$$

At the upper wall, the potential function is

$$\varphi(x, f_-(x)) = \Phi_-(x), \quad l_- \leq x \leq 0$$

satisfying $\Phi_-(l_-) = \zeta_-$, $\Phi_-(0) = 0$ and

$$\Phi'_-(x) = \frac{\mathcal{Q}_-(x)}{\cos \Theta_-(x)} > 0, \quad l_- \leq x \leq 0.$$

Thus, Φ_- is expressed as

$$\Phi_-(x) = \int_0^x \frac{\mathcal{Q}_-(s)}{\cos \Theta_-(s)} ds, \quad l_- \leq x \leq 0. \quad (3.29)$$

Denote by X_- the inverse function of Φ_- , i.e.

$$X_-(\varphi) = \Phi_-^{-1}(\varphi), \quad \zeta_- \leq \varphi \leq 0. \quad (3.30)$$

Therefore, the subsonic-sonic flow problem in the potential plane is formulated as follows

$$\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}), \quad (3.31)$$

$$\frac{\partial A(q)}{\partial \varphi}(\zeta_-, \psi) = - \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)}, \quad \psi \in (0, m_{\text{in}}), \quad (3.32)$$

$$\frac{\partial q}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (\zeta_-, 0), \quad (3.33)$$

$$\frac{\partial B(q)}{\partial \psi}(\varphi, m_{\text{in}}) = \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)}, \quad \varphi \in (\zeta_-, 0), \quad (3.34)$$

$$q(0, \psi) = c_*, \quad \psi \in (0, m_{\text{in}}), \quad (3.35)$$

$$\mathcal{Q}_{\text{in}}(y) = q(0, \psi) \Big|_{\psi=\Psi_{\text{in}}(y)}, \quad y \in [0, f_-(l_-)], \quad (3.36)$$

$$\mathcal{Q}_-(x) = q(\varphi, m_{\text{in}}) \Big|_{\varphi=\Phi_-(x)}, \quad x \in [l_-, 0]. \quad (3.37)$$

Remark 3.2 A natural boundary condition at the inlet seems to be

$$\frac{\partial A(q)}{\partial \varphi}(\zeta_-, \psi) = - \frac{\Theta'_{\text{in}}(Y_{\text{in}}(\psi)) \cos \Theta_{\text{in}}(Y_{\text{in}}(\psi))}{q(\zeta_-, \psi) \rho(q^2(\zeta_-, \psi))}, \quad \psi \in (0, m_{\text{in}}) \quad (3.38)$$

instead of (3.32). However, since $\frac{d}{dq} \left(\frac{1}{q \rho(q^2)} \right) < 0$ in $q \in (0, c_*)$ and $-\Theta'_{\text{in}}(Y_{\text{in}}(\psi)) \cos \Theta_{\text{in}}(Y_{\text{in}}(\psi)) \geq 1/(2R_0)$ from (3.24), it seems difficult to obtain the well-posedness of the problem (3.31), (3.38), (3.33)–(3.35). Therefore, we prescribe the boundary condition at the inlet by (3.32).

The problem (3.31)–(3.37) is a boundary value problem for a second-order quasilinear degenerate equation in a rectangle with two free parameters. Furthermore, the degeneracy is characteristic and the boundary conditions (3.32) and (3.34) are nonlinear, nonlocal and implicit. As far as we know, there is no known theory to show the existence and uniqueness of solutions even if the problem is uniformly elliptic. Thus we will use a fixed point argument to prove the existence of solutions to the problem (3.31)–(3.37) as follows: For given \mathcal{Q}_{in} and \mathcal{Q}_- , we solve the problem (3.31)–(3.35), then define a mapping by (3.36) and (3.37). To this end, one must determine and control precisely the rate of the solution tending to c_* in order to solve the problem (3.31)–(3.37). However, there is no a background solution to this problem to suggest what the rate is. And we will carry out some precise elliptic estimates in this paper to determine and control this rate.

Since (3.31) is degenerate, it is more convenient to introduce weak solutions to the problem (3.31)–(3.35). After determining the rate of the solution tending to c_* , one can establish the regularity of weak solutions.

Definition 3.1 *A function $q \in L^\infty((\zeta_-, 0) \times (0, m_{\text{in}}))$ is said to be a weak solution to the problem (3.31)–(3.35), if*

$$0 < \inf_{(\zeta_-, 0) \times (0, m_{\text{in}})} q \leq \sup_{(\zeta_-, 0) \times (0, m_{\text{in}})} q \leq c_*$$

and

$$\begin{aligned} & \int_{\zeta_-}^0 \int_0^{m_{\text{in}}} \left(A(q(\varphi, \psi)) \frac{\partial^2 \xi}{\partial \varphi^2}(\varphi, \psi) + B(q(\varphi, \psi)) \frac{\partial^2 \xi}{\partial \psi^2}(\varphi, \psi) \right) d\varphi d\psi \\ & + \int_0^{m_{\text{in}}} \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} \xi(\zeta_-, \psi) d\psi + \int_{\zeta_-}^0 \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)} \xi(\varphi, m_{\text{in}}) d\varphi = 0 \end{aligned}$$

for any $\xi \in C^2([\zeta_-, 0] \times [0, m_{\text{in}}])$ with

$$\frac{\partial \xi}{\partial \psi}(\cdot, 0) \Big|_{(\zeta_-, 0)} = \frac{\partial \xi}{\partial \psi}(\cdot, m_{\text{in}}) \Big|_{(\zeta_-, 0)} = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \varphi}(\zeta_-, \cdot) \Big|_{(0, m_{\text{in}})} = \xi(0, \cdot) \Big|_{(0, m_{\text{in}})} = 0.$$

Though the condition (3.22) seems quite strict, yet, it is almost necessary for the existence of C^2 solutions as shown in the following remark.

Remark 3.3 *If the problem (3.31)–(3.35) admits a $C^2([\zeta_-, 0] \times [0, m_{\text{in}}])$ solution, then*

$$f''_-(x) = O(x^2), \quad l_- < x < 0. \quad (3.39)$$

Proof. It follows from (3.31) and (3.35) that

$$A''(c_*) \left(\frac{\partial q}{\partial \varphi}(0, \psi) \right)^2 + B'(c_*) \frac{\partial^2 q}{\partial \psi^2}(0, \psi) + B''(c_*) \left(\frac{\partial q}{\partial \psi}(0, \psi) \right)^2 = 0, \quad 0 \leq \psi \leq m_{\text{in}}$$

and

$$\frac{\partial q}{\partial \psi}(0, \psi) = 0, \quad \frac{\partial^2 q}{\partial \psi^2}(0, \psi) = 0, \quad 0 \leq \psi \leq m_{\text{in}},$$

which imply

$$\frac{\partial q}{\partial \varphi}(0, \psi) = 0, \quad 0 \leq \psi \leq m_{\text{in}}$$

and thus

$$\frac{\partial q}{\partial \varphi}(\varphi, \psi) = O(\varphi), \quad q(\varphi, \psi) = c_* - O(\varphi^2), \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}).$$

Hence

$$\begin{aligned} \frac{\partial^2 A(q)}{\partial \varphi^2}(\varphi, \psi) &= A'(q(\varphi, \psi)) \frac{\partial^2 q}{\partial \varphi^2}(\varphi, \psi) + A''(q(\varphi, \psi)) \left(\frac{\partial q}{\partial \varphi}(\varphi, \psi) \right)^2 = O(\varphi^2), \\ &(\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}), \end{aligned}$$

which, together with (3.31), (3.33) and (3.34), yields

$$\frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)} = - \int_0^{m_{\text{in}}} \frac{\partial^2 A(q)}{\partial \varphi^2}(\varphi, \psi) d\psi = O(\varphi^2), \quad \varphi \in (\zeta_-, 0). \quad (3.40)$$

Then, (3.39) follows from (3.40), $\Theta_-(0) = 0$, $\mathcal{Q}_-(0) = c_* > 0$, $X_-(0) = 0$ and $X'_-(0) = 1/c_* > 0$. \square

3.3 Formulation of the sonic-supersonic flow problem in the potential plane

A sonic-supersonic flow is expected in the divergent part of the nozzle Ω_+ . In this subsection, we formulate this sonic-supersonic flow problem in the potential plane.

Assume that $f_+ \in C^3([0, l_+])$ ($0 < l_+ \leq 1$) satisfies

$$\delta_{1,+} x^{\lambda_+} \leq f_+''(x) \leq \delta_{2,+} x^{\lambda_+}, \quad f_+'''(x) \geq 0, \quad 0 \leq x \leq l_+ \quad (3.41)$$

with positive constants λ_+ , $\delta_{1,+}$ and $\delta_{2,+}$ such that $\lambda_+ > 2$ and $\delta_{1,+} \leq \delta_{2,+}$.

Remark 3.4 *The second condition in (3.41) can be relaxed by*

$$f_+'''(x) \geq -\omega x^{\lambda_+-1}, \quad 0 \leq x \leq l_+,$$

where ω is a positive constant depending only on γ , m , λ_+ , $\delta_{1,+}$ and $\delta_{2,+}$ (see Remark 5.2 for the detail).

Set

$$Q(\varphi, \psi) = A(q(\varphi, \psi)).$$

Then, (2.3) takes the form

$$Q_{\varphi\varphi} + (K'(Q)Q_{\psi})_{\psi} = 0, \quad (3.42)$$

where

$$K(s) = B(A_+^{-1}(s)), \quad s < 0$$

and

$$K'(s) = \left(1 - \frac{\gamma+1}{2}q^2\right)^{-1} \left(1 - \frac{\gamma-1}{2}q^2\right)^{2/(\gamma-1)+1} \Big|_{q=A_+^{-1}(s)} < 0, \quad s < 0,$$

$$K''(s) = (\gamma+1)q^4 \left(1 - \frac{\gamma+1}{2}q^2\right)^{-3} \left(1 - \frac{\gamma-1}{2}q^2\right)^{3/(\gamma-1)+1} \Big|_{q=A_+^{-1}(s)} < 0, \quad s < 0.$$

Since $\lim_{s \rightarrow 0^-} K'(s) = -\infty$, (3.42) is a singular nonlinear wave equation in the sonic-supersonic region. We now transform (3.42) into a first order system to use the method of characteristics. Set

$$U(\varphi, \psi) = Q_\varphi(\varphi, \psi), \quad V(\varphi, \psi) = Q_\psi(\varphi, \psi).$$

Then, (3.42) is transformed into the following system

$$\begin{aligned} U_\varphi - (b(Q)V)_\psi &= 0, \\ V_\varphi - U_\psi &= 0, \end{aligned}$$

i.e.

$$\begin{pmatrix} U \\ V \end{pmatrix}_\varphi + \begin{pmatrix} 0 & -b(Q) \\ -1 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_\psi + \begin{pmatrix} 0 & -p(Q)Q_\psi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (3.43)$$

where

$$b(Q) = -K'(Q) > 0, \quad p(Q) = -K''(Q) > 0, \quad Q < 0.$$

Set

$$R(Q) = -\frac{1}{2\sqrt{b(Q)}} \begin{pmatrix} -\sqrt{b(Q)} & \sqrt{b(Q)} \\ 1 & 1 \end{pmatrix}, \quad R^{-1}(Q) = \begin{pmatrix} 1 & -\sqrt{b(Q)} \\ -1 & -\sqrt{b(Q)} \end{pmatrix}.$$

Multiplying (3.43) by $R^{-1}(Q)$ on the left side, one gets

$$R^{-1}(Q) \begin{pmatrix} U \\ V \end{pmatrix}_\varphi + \begin{pmatrix} \sqrt{b(Q)} & 0 \\ 0 & -\sqrt{b(Q)} \end{pmatrix} R^{-1}(Q) \begin{pmatrix} U \\ V \end{pmatrix}_\psi + R^{-1}(Q) \begin{pmatrix} 0 & -p(Q)Q_\psi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$\begin{aligned} \begin{pmatrix} W \\ Z \end{pmatrix}_\varphi + \begin{pmatrix} \sqrt{b(Q)} & 0 \\ 0 & -\sqrt{b(Q)} \end{pmatrix} \begin{pmatrix} W \\ Z \end{pmatrix}_\psi &= -R^{-1}(Q) \begin{pmatrix} 0 & -p(Q)Q_\psi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \frac{\partial R^{-1}(Q)}{\partial \varphi} \begin{pmatrix} U \\ V \end{pmatrix} \\ &+ \begin{pmatrix} \sqrt{b(Q)} & 0 \\ 0 & -\sqrt{b(Q)} \end{pmatrix} \frac{\partial R^{-1}(Q)}{\partial \psi} \begin{pmatrix} U \\ V \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} W \\ Z \end{pmatrix} = R^{-1}(Q) \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U - \sqrt{b(Q)}V \\ -U - \sqrt{b(Q)}V \end{pmatrix} = \begin{pmatrix} Q_\varphi - \sqrt{b(Q)}Q_\psi \\ -Q_\varphi - \sqrt{b(Q)}Q_\psi \end{pmatrix}.$$

Therefore, (3.43) becomes

$$\begin{aligned} W_\varphi + b^{1/2}(Q)W_\psi &= \frac{1}{4}b^{-1}(Q)p(Q)(Q_\varphi - b^{1/2}(Q)Q_\psi)(W + Z), \\ Z_\varphi - b^{1/2}(Q)Z_\psi &= \frac{1}{4}b^{-1}(Q)p(Q)(Q_\varphi + b^{1/2}(Q)Q_\psi)(W + Z). \end{aligned}$$

The flow angle on the upper wall is

$$\Theta_+(x) = \arctan f'_+(x), \quad 0 \leq x \leq l_+.$$

Let the speed of the flow on the upper wall be given by

$$q(x, f_+(x)) = \mathcal{Q}_+(x), \quad 0 \leq x \leq l_+.$$

Then, the potential function on the upper wall is

$$\varphi(x, f_+(x)) = \Phi_+(x), \quad 0 \leq x \leq l_+$$

satisfying $\Phi_+(0) = 0$ and

$$\Phi'_+(x) = \frac{\mathcal{Q}_+(x)}{\cos \Theta_+(x)} > 0, \quad 0 \leq x \leq l_+.$$

Thus, Φ_+ is expressed as

$$\Phi_+(x) = \int_0^x \frac{\mathcal{Q}_+(s)}{\cos \Theta_+(s)} ds, \quad 0 \leq x \leq l_+. \quad (3.44)$$

Denote by X_+ the inverse function of Φ_+ , i.e.

$$X_+(\varphi) = \Phi_+^{-1}(\varphi), \quad 0 \leq \varphi \leq \zeta_+, \quad (3.45)$$

where

$$\zeta_+ = \Phi_+(l_+). \quad (3.46)$$

Therefore, the sonic-supersonic flow problem in the potential plane can be formulated as

$$Q_{\varphi\varphi} - (b(Q)Q_{\psi})_{\psi} = 0, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (3.47)$$

$$Q(0, \psi) = 0, \quad \psi \in (0, m), \quad (3.48)$$

$$Q_{\varphi}(0, \psi) = 0, \quad \psi \in (0, m), \quad (3.49)$$

$$Q_{\psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (3.50)$$

$$Q_{\psi}(\varphi, m) = -\frac{\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{b(Q(\varphi, m))A_+^{-1}(Q(\varphi, m))}, \quad \varphi \in (0, \zeta_+), \quad (3.51)$$

$$\mathcal{Q}_+(x) = A_+^{-1}(Q(\Phi_+(x), m)) \quad x \in [0, l_+] \quad (3.52)$$

or equivalently as

$$W_{\varphi} + b^{1/2}(Q)W_{\psi} = \frac{1}{4}b^{-1}(Q)p(Q)(Q_{\varphi} - b^{1/2}(Q)Q_{\psi})(W + Z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (3.53)$$

$$Z_{\varphi} - b^{1/2}(Q)Z_{\psi} = \frac{1}{4}b^{-1}(Q)p(Q)(Q_{\varphi} + b^{1/2}(Q)Q_{\psi})(W + Z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (3.54)$$

$$W(0, \psi) = 0, \quad \psi \in (0, m), \quad (3.55)$$

$$Z(0, \psi) = 0, \quad \psi \in (0, m), \quad (3.56)$$

$$W(\varphi, 0) + Z(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (3.57)$$

$$W(\varphi, m) + Z(\varphi, m) = \frac{2\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{b^{1/2}(Q(\varphi, m))A_+^{-1}(Q(\varphi, m))}, \quad \varphi \in (0, \zeta_+), \quad (3.58)$$

$$Q_{\varphi}(\varphi, \psi) = \frac{1}{2}(W(\varphi, \psi) - Z(\varphi, \psi)), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (3.59)$$

$$Q(0, \psi) = 0, \quad \psi \in (0, m), \quad (3.60)$$

$$\mathcal{Q}_+(x) = A_+^{-1}(Q(\Phi_+(x), m)), \quad x \in [0, l_+]. \quad (3.61)$$

The problems (3.47)–(3.51) and (3.53)–(3.60) should be regarded as initial boundary value problems, where φ -direction plays the role of the time. Since $\lim_{Q \rightarrow 0^-} b(Q) = +\infty$, (3.47), (3.53) and (3.54) are singular at sonic points. Furthermore, as shown in Theorem 2.5, all positive and negative characteristics from any sonic point are the same line $\{0\} \times [0, m]$. That is to say, the singularity at the sonic curve is so strong in the sense that there is not any characteristic moving from a sonic point to the supersonic region. The source terms in (3.53) and (3.54) are also singular at sonic points because $\lim_{Q \rightarrow 0^-} b^{-1}(Q)p(Q) = +\infty$. Moreover, the boundary conditions (3.51) and (3.58) are nonlinear, nonlocal and implicit. We will use a fixed point argument to prove the existence of solutions to the problem (3.53)–(3.61) as follows: For given Q and \mathcal{Q}_- , we first solve the problem (3.53)–(3.58), then define a new Q by solving the problem (3.59), (3.60) and a new \mathcal{Q}_- by (3.61). To get a fixed point, one must choose a space with a precise behavior near the sonic curve for Q and then show that the new Q also belongs to this space by some elaborate calculations and optimal estimates.

Since (3.47), (3.53) and (3.54) are singular, we first seek weak solutions, which are defined as follows. By more complicated and precise estimates, one can get smooth solutions.

Definition 3.2 A function Q is said to be a weak solution to the problem (3.47)–(3.51) if $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ satisfies (3.48) and

$$\begin{aligned} & \int_0^{\zeta_+} \int_0^m (Q_\varphi(\varphi, \psi) \xi_\varphi(\varphi, \psi) - b(Q(\varphi, \psi)) Q_\psi(\varphi, \psi) \xi_\psi(\varphi, \psi)) d\varphi d\psi \\ & - \int_0^{\zeta_+} \frac{\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{A_+^{-1}(Q(\varphi, m))} \xi(\varphi, m) d\varphi = 0 \end{aligned}$$

for any $\xi \in C^1([0, \zeta_+] \times [0, m])$ with $\xi(\zeta_+, \cdot)|_{(0, m)} = 0$.

Definition 3.3 A triad of functions (W, Z, Q) is said to be a weak solution to the problem (3.53)–(3.60) if $W, Z \in L^\infty((0, \zeta_+) \times (0, m))$ and $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ satisfy

(i) For any $\xi, \eta \in C^1([0, \zeta_+] \times [0, m])$ with

$$\xi(\zeta_+, \cdot)|_{(0, m)} = \eta(\zeta_+, \cdot)|_{(0, m)} = 0, \quad \xi(\cdot, 0)|_{(0, \zeta_+)} = -\eta(\cdot, 0)|_{(0, \zeta_+)}, \quad \xi(\cdot, m)|_{(0, \zeta_+)} = -\eta(\cdot, m)|_{(0, \zeta_+)},$$

it holds that

$$\begin{aligned} & \int_0^{\zeta_+} \int_0^m \left(W \xi_\varphi + b^{1/2}(Q) W \xi_\psi + \frac{1}{2} b^{-1/2}(Q) p(Q) Q_\psi W \xi \right. \\ & \quad \left. + \frac{1}{4} b^{-1}(Q) p(Q) (Q_\varphi - b^{1/2}(Q) Q_\psi) (W + Z) \xi \right) d\varphi d\psi \\ & + \int_0^{\zeta_+} \int_0^m \left(Z \eta_\varphi - b^{1/2}(Q) Z \eta_\psi - \frac{1}{2} b^{-1/2}(Q) p(Q) Q_\psi Z \eta \right. \\ & \quad \left. + \frac{1}{4} b^{-1}(Q) p(Q) (Q_\varphi + b^{1/2}(Q) Q_\psi) (W + Z) \eta \right) d\varphi d\psi \\ & - 2 \int_0^{\zeta_+} \frac{\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{A_+^{-1}(Q(\varphi, m))} \xi(\varphi, m) d\varphi = 0; \end{aligned}$$

- (ii) (3.59) holds in the sense of distribution;
- (iii) Q satisfies (3.60).

As the discussion for the subsonic-sonic flow problem in Remark 3.3, the condition (3.41) on the wall of the nozzle is almost necessary for the existence of C^2 solutions.

Remark 3.5 *If the problem (3.47)–(3.51) admits a $C^2([0, \zeta_+] \times [0, m])$ solution, then*

$$f_+''(x) = O(x^2), \quad 0 < x < l_+.$$

3.4 Main results of the transonic flow problem in the de Laval nozzle

We now state the main results on the transonic flow problem. Since the sonic curve of the transonic flow studied in the paper is located at the throat of the nozzle, the results of the transonic flow can be stated for the subsonic-sonic part and the sonic-supersonic part separately.

For the subsonic-sonic flow in the convergent part of the nozzle, the first result concerns the existence and regularity estimates of such a solution.

Theorem 3.1 *Assume that $f_- \in C^3([l_-, 0])$ satisfies (3.22) and $g \in C^3([0, f_-(l_-)])$ satisfies (3.23) and (3.24). There exists a positive constant $\delta_{0,-}$ depending only on $\gamma, m, \lambda_-, \delta_{1,-}, \delta_{2,-}$ and $\epsilon(\cdot)$, such that for any $-\delta_{0,-} \leq l_- < 0$, the problem (3.31)–(3.37) admits a weak solution $q \in C^\infty((\zeta_-, 0) \times (0, m_{\text{in}})) \cap C^1([\zeta_-, 0] \times [0, m_{\text{in}}]) \cap C([\zeta_-, 0] \times [0, m_{\text{in}}])$ with $m_{\text{in}} = m$ satisfying*

$$\left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq C_{1,-}(-\zeta_-)^{\lambda_-/2}(-\varphi)^{1/2}, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m), \quad (3.62)$$

$$\left| A(q(\varphi', \psi')) - A(q(\varphi'', \psi'')) \right| \leq C_{1,-}(|\varphi' - \varphi''|^{1/2} + |\psi' - \psi''|), \quad (\varphi', \psi'), (\varphi'', \psi'') \in (\zeta_-, 0) \times (0, m), \quad (3.63)$$

$$c_* - C_{3,-}(-\varphi)^{\lambda_-/2+1} \leq q(\varphi, \psi) \leq c_* - C_{2,-}(-\varphi)^{\lambda_-/2+1}, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m), \quad (3.64)$$

where $C_{i,-}$ ($i = 1, 2, 3$) ($C_{2,-} \leq C_{3,-}$) are positive constants depending only on $\gamma, m, \lambda_-, \delta_{1,-}$ and $\delta_{2,-}$. Moreover,

- (i) *If f_- satisfies*

$$|f_-'''(x)| \leq \delta_{3,-}(-x)^{\lambda_-/4+1/2}, \quad l_- \leq x \leq 0 \quad (3.65)$$

with a positive constant $\delta_{3,-}$ additionally, then any weak solution satisfies $q \in C^1([\zeta_-, 0] \times [0, m_{\text{in}}])$ and

$$\left| \frac{\partial q}{\partial \varphi}(\varphi, \psi) \right| \leq C_{4,-}(-\varphi)^{\lambda_-/4+1/2}, \quad \left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq C_{4,-}(-\varphi)^{\lambda_-/2+1}, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m), \quad (3.66)$$

where the positive constant $C_{4,-}$ depends only on $\gamma, m, \lambda_-, \delta_{1,-}, \delta_{2,-}$ and $\delta_{3,-}$.

- (ii) *If $f_- \in C^4([l_-, 0])$ satisfies (3.65) and*

$$|f_-^{(4)}(x)| \leq \delta_{4,-}, \quad l_- \leq x \leq 0 \quad (3.67)$$

with a positive constant $\delta_{4,-}$ additionally, then any weak solution satisfies $q \in C^{1,1}((\zeta_-, 0] \times [0, m])$ and

$$\left| \frac{\partial^2 q}{\partial \varphi^2}(\varphi, \psi) \right| \leq C_{5,-}, \quad \left| \frac{\partial^2 q}{\partial \varphi \partial \psi}(\varphi, \psi) \right| \leq C_{5,-}(-\varphi)^{\lambda_-/4+1/2}, \quad \left| \frac{\partial^2 q}{\partial \psi^2}(\varphi, \psi) \right| \leq C_{5,-}(-\varphi)^{\lambda_-/2+1},$$

$$(\varphi, \psi) \in (\zeta_-/2, 0) \times (0, m) \quad (3.68)$$

with a positive constant $C_{5,-}$ depending only on $\gamma, m, \lambda_-, \delta_{1,-}, \delta_{2,-}, \delta_{3,-}$ and $\delta_{4,-}$.

(iii) If $g \in C^{3,\alpha}([0, f_-(l_-)])$ and $f_- \in C^{3,\alpha}([l_-, 0])$ for a number $\alpha \in (0, 1)$ additionally, then any weak solution satisfies $q \in C^{2,\alpha}([\zeta_-, 0] \times [0, m])$.

The next result yields the uniqueness.

Theorem 3.2 *There exists at most one solution $\varphi \in C^{1,1}(\overline{\Omega_-})$ to the problem (3.10)–(3.15).*

For the sonic-supersonic flow in the divergent part of the nozzle, we start with the existence and regularity estimates.

Theorem 3.3 *Assume that $f_+ \in C^3([0, l_+])$ satisfies (3.41). There exists a positive constant $\delta_{0,+} > 0$ depending only on $\gamma, m, \lambda_+, \delta_{1,+}$ and $\delta_{2,+}$, such that for any $0 < l_+ \leq \delta_{0,+}$, the problem (3.47)–(3.52) admits at least one weak solution $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ satisfying*

$$-C_{2,+}\varphi^{\lambda_++2} \leq Q(\varphi, \psi) \leq -C_{1,+}\varphi^{\lambda_++2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (3.69)$$

$$-C_{2,+}\varphi^{\lambda_++1} \leq Q_\varphi(\varphi, \psi) \leq -C_{1,+}\varphi^{\lambda_++1}, \quad |Q_\psi(\varphi, \psi)| \leq C_{2,+}\varphi^{3\lambda_+/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \quad (3.70)$$

with positive constants $C_{1,+}$ and $C_{2,+}$ ($C_{1,+} \leq C_{2,+}$) depending only on $\gamma, m, \lambda_+, \delta_{1,+}$ and $\delta_{2,+}$. Furthermore, assume that f_+ satisfies

$$|f_+'''(x)| \leq \delta_{3,+}x^{\lambda_+-1}, \quad 0 \leq x \leq l_+ \quad (3.71)$$

with a positive constant $\delta_{3,+}$ additionally. Then there exist positive constants $\tilde{\delta}_{0,+}$ and $C_{3,+}$, both depending only on $\gamma, m, \lambda_+, \delta_{1,+}, \delta_{2,+}$ and $\delta_{3,+}$, such that if $0 < l_+ \leq \tilde{\delta}_{0,+}$, then the problem (3.47)–(3.52) admits at least a solution $Q \in C^{1,1}([0, \zeta_+] \times [0, m])$, which satisfies (3.69), (3.70) and

$$|Q_{\varphi\varphi}(\varphi, \psi)| \leq C_{3,+}\varphi^{\lambda_+}, \quad |Q_{\varphi\psi}(\varphi, \psi)| \leq C_{3,+}\varphi^{5\lambda_+/4+1/2}, \quad |Q_{\psi\psi}(\varphi, \psi)| \leq C_{3,+}\varphi^{3\lambda_+/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (3.72)$$

The following theorem shows the uniqueness of weak solutions.

Theorem 3.4 *Assume that $f_+ \in C^4([0, l_+])$ satisfies (3.41) and (3.71). Then the problem (3.47)–(3.52) admits at most one weak solution $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ satisfying (3.69) and (3.70) with any given positive constants $C_{i,+}$ ($i = 1, 2$).*

Let us connect the subsonic-sonic flow and the sonic-supersonic flow to get a global smooth transonic flow.

Theorem 3.5 *Assume that $f_- \in C^4([l_-, 0])$ satisfies (3.22), (3.65) and (3.67), $g \in C^{3,\alpha}([0, f_-(l_-)])$ satisfies (3.23) and (3.24), while $f_+ \in C^4([0, l_+])$ satisfies (3.41) and (3.71). If $-l_-$ and l_+ are sufficiently small, the problem (3.31)–(3.37) admits a unique solution $q_- \in C^{1,1}([\zeta_-, 0] \times [0, m])$ satisfying (3.62)–(3.64), (3.66) and (3.68), while the problem (3.47)–(3.52) admits a unique solution*

$Q_+ \in C^{1,1}([0, \zeta_+] \times [0, m])$ satisfying (3.69), (3.70) and (3.72). Connect q_- and Q_+ in the following way

$$q(\varphi, \psi) = \begin{cases} q_-(\varphi, \psi), & (\varphi, \psi) \in [\zeta_-, 0] \times [0, m], \\ A_+^{-1}(Q_+(\varphi, \psi)), & (\varphi, \psi) \in [0, \zeta_+] \times [0, m]. \end{cases}$$

Then, $q \in C^{1,1}([\zeta_-, \zeta_+] \times [0, m])$ is a solution to (2.3). Furthermore, $\frac{\partial q}{\partial \varphi} = \frac{\partial q}{\partial \psi} = 0$ at the sonic curve and $\frac{\partial^2 q}{\partial \psi^2}$ is continuous at sonic points. Therefore, q satisfies (2.3) at the sonic curve, i.e.

$$\lim_{\varphi \rightarrow 0} \left(\frac{\partial^2 A(q(\varphi, \psi))}{\partial \varphi^2} + \frac{\partial^2 B(q(\varphi, \psi))}{\partial \psi^2} \right) = 0, \quad 0 \leq \psi \leq m.$$

In the physical plane, Theorem 3.5 can be stated as follows.

Theorem 3.6 Assume that $f_- \in C^4([l_-, 0])$ satisfies (3.22), (3.65) and (3.67), $g \in C^{3,\alpha}([0, f_-(l_-)])$ satisfies (3.23) and (3.24), while $f_+ \in C^4([0, l_+])$ satisfies (3.41) and (3.71). If $-l_-$ and l_+ are sufficiently small, the transonic flow problem (3.2)–(3.9) admits a unique classical solution $\varphi \in C^{2,1}(\overline{\Omega})$, which satisfies (3.62)–(3.64), (3.66) and (3.68) for $q = |\nabla \varphi|$ in Ω_- , while satisfies (3.69), (3.70) and (3.72) for $Q = A(|\nabla \varphi|)$ in Ω_+ .

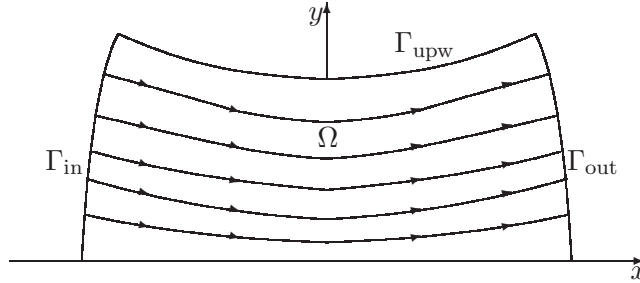


Figure: the smooth transonic flow in the de Laval nozzle

Remark 3.6 In Theorem 3.6 we get a $C^{1,1}$ transonic flow. However, this transonic flow pattern is strongly singular in the sense that the sonic curve is a characteristic degenerate boundary in the subsonic-sonic region, while in the sonic-supersonic region all characteristics from sonic points coincide, which are the sonic curve and never approach the supersonic region.

Remark 3.7 In the de Laval nozzle Ω , there is also a subsonic-sonic-subsonic flow, whose sonic curve is located at the throat of the nozzle.

Remark 3.8 For the nozzle Ω , if there is a flat part between the convergent part and the divergent part, there is also a transonic flow, which is subsonic in the convergent part, sonic in the flat part, while supersonic in the divergent part.

Remark 3.9 The geometry of nozzles is important for smooth transonic flows. As mentioned in Remarks 3.3 and 3.5,

$$f''(x) = O(x^2), \quad l_- < x < l_+$$

is necessary for a C^2 transonic flow whose sonic curve is located at the throat of the nozzle. Moreover, in [20], we get a continuous subsonic-sonic flow in a convergent nozzle with straight solid walls. However, this subsonic-sonic flow is singular in the sense that while the speed is continuous yet the acceleration blows up at the sonic state, and there is no way to extend it to be a transonic flow or a subsonic-sonic-subsonic flow.

4 Subsonic-sonic flows in the convergent part of the nozzle

In this section, we will establish the well-posedness of the subsonic-sonic flow problem in the convergent part of the nozzle. For convenience, we abbreviate λ_- and $\delta_{k,-}$ by λ and δ_k for $k = 1, 2, 3, 4$, respectively, and use $C_i, \underline{c}, \mu_i, \kappa_i, M_i$ to denote generic positive constants. Furthermore, a parenthesis after a generic constant means that this constant depends only on the variables in the parentheses.

Owing to (3.22), $f_-(l_-)$ and R_0 satisfy

$$mc_*^{-1-2/(\gamma-1)} < f_-(l_-) < mc_*^{-1-2/(\gamma-1)} + \delta_2, \quad -\delta_2 < f'_-(l_-) < 0$$

and

$$\frac{\delta_1 c_*^{1+2/(\gamma-1)}}{(\lambda+1)(m+\delta_2 c_*^{1+2/(\gamma-1)})\sqrt{\delta_2^2+1}}(-l_-)^{\lambda+1} < \frac{1}{R_0} < \frac{\delta_2 c_*^{1+2/(\gamma-1)}}{(\lambda+1)m}(-l_-)^{\lambda+1}$$

with m given by (3.1).

In order to solve the problem (3.31)–(3.35) by a fixed point argument, one needs to specify \mathcal{Q}_{in} and \mathcal{Q}_- in advance as follows: $\mathcal{Q}_{\text{in}} \in C^{0,1}([0, f_-(l_-)])$ satisfies

$$\max \left\{ \frac{c_*}{2}, c_* - C_2(-l_-)^{\lambda/2+1} \right\} \leq \mathcal{Q}_{\text{in}}(y) \leq \max \left\{ \frac{c_*}{2}, c_* - C_1(-l_-)^{\lambda/2+1} \right\}, \quad |\mathcal{Q}'_{\text{in}}(y)| \leq 1, \\ 0 \leq y \leq f_-(l_-) \quad (4.1)$$

and $\mathcal{Q}_- \in C([l_-, 0]) \cap C^{1/2}([l_-, l_-/2])$ satisfies

$$\underline{c} \leq \mathcal{Q}_-(x) \leq c_*, \quad l_- \leq x \leq 0, \quad [\mathcal{Q}_-]_{1/2; (l_-, l_-/2)} \leq C_0, \quad (4.2)$$

where C_i ($i = 0, 1, 2$) and \underline{c} ($C_1 \leq C_2$, $\underline{c} < c_*$) will be determined. Then, $m_{\text{in}}, \zeta_-, \Psi'_{\text{in}}$ and Φ'_- satisfy

$$m_1 = \frac{1}{2}mc_*^{-2/(\gamma-1)}\rho(c_*^2/4) \leq m_{\text{in}} \leq m_2 = c_*\rho(c_*^2)(mc_*^{-1-2/(\gamma-1)} + \delta_2)\sqrt{\delta_2^2+1},$$

$$c_*\sqrt{\delta_2^2+1}l_- \leq \zeta_- \leq \underline{c}l_-,$$

$$\frac{1}{2}c_*\rho(c_*^2/4) \leq \Psi'_{\text{in}}(y) \leq c_*\rho(c_*^2)\sqrt{\delta_2^2+1}, \quad 0 \leq y \leq f_-(l_-),$$

$$\underline{c} \leq \Phi'_-(x) \leq c_*\sqrt{\delta_2^2+1}, \quad l_- \leq x \leq 0.$$

Moreover, one can verify that

$$\mu_1(-\zeta_-)^{\lambda+1} \leq -\frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} \leq \mu_2(-\zeta_-)^{\lambda+1}, \quad \psi \in (0, m_{\text{in}}), \quad (4.3)$$

$$\left| \left(\frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} \right)' \right| \leq \mu_2(C_2 + 1)(\epsilon(\zeta_-) + \zeta_-^2)(-\zeta_-)^{3\lambda/2}, \quad \psi \in (0, m_{\text{in}}), \quad (4.4)$$

$$\mu_1(-\varphi)^\lambda \leq \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)} \leq \mu_2(-\varphi)^\lambda, \quad \varphi \in (\zeta_-, 0), \quad (4.5)$$

where μ_1 and μ_2 ($\mu_1 \leq \mu_2$) depend only on γ , m , λ , δ_1 , δ_2 and \underline{c} .

4.1 A comparison principle

We will state a comparison principle for weak solutions to the problem (3.31)–(3.35). More generally, instead of (3.35), one imposes the following boundary condition

$$q(0, \psi) = c, \quad \psi \in (0, m_{\text{in}}), \quad (4.6)$$

where $0 < c \leq c_*$. Weak solutions, supersolutions and subsolutions to the problem (3.31)–(3.34), (4.6) are defined in the following sense.

Definition 4.1 *A function $q \in L^\infty((\zeta_-, 0) \times (0, m_{\text{in}}))$ is said to be a weak supersolution (subsolution) to the problem (3.31)–(3.34), (4.6), if*

$$0 < \inf_{(\zeta_-, 0) \times (0, m_{\text{in}})} q \leq \sup_{(\zeta_-, 0) \times (0, m_{\text{in}})} q \leq c_*$$

and

$$\begin{aligned} & \int_{\zeta_-}^0 \int_0^{m_{\text{in}}} \left(A(q(\varphi, \psi)) \frac{\partial^2 \xi}{\partial \varphi^2}(\varphi, \psi) + B(q(\varphi, \psi)) \frac{\partial^2 \xi}{\partial \psi^2}(\varphi, \psi) \right) d\varphi d\psi \\ & - A(c) \int_0^{m_{\text{in}}} \frac{\partial \xi}{\partial \varphi}(0, \psi) d\psi + \int_0^{m_{\text{in}}} \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} \xi(\zeta_-, \psi) d\psi \\ & + \int_{\zeta_-}^0 \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)} \xi(\varphi, m_{\text{in}}) d\varphi \leq (\geq) 0 \end{aligned}$$

for any nonnegative function $\xi \in C^2([\zeta_-, 0] \times [0, m_{\text{in}}])$ with

$$\frac{\partial \xi}{\partial \psi}(\cdot, 0) \Big|_{(\zeta_-, 0)} = \frac{\partial \xi}{\partial \psi}(\cdot, m_{\text{in}}) \Big|_{(\zeta_-, 0)} = 0 \quad \text{and} \quad \frac{\partial \xi}{\partial \varphi}(\zeta_-, \cdot) \Big|_{(0, m_{\text{in}})} = \xi(0, \cdot) \Big|_{(0, m_{\text{in}})} = 0.$$

Furthermore, $q \in L^\infty((\zeta_-, 0) \times (0, m_{\text{in}}))$ is said to be a weak solution to the problem (3.31)–(3.34), (4.6) if q is both a weak supersolution and a weak subsolution.

Let $q \in L^\infty((\zeta_-, 0) \times (0, m_{\text{in}}))$ be a weak solution to the problem (3.31)–(3.34), (4.6). If q satisfies

$$\sup_{(\zeta_-, 0) \times (0, m_{\text{in}})} q < c_*, \quad (4.7)$$

then $q \in C^\infty((\zeta_-, 0) \times (0, m_{\text{in}})) \cap C^1([\zeta_-, 0] \times [0, m_{\text{in}}]) \cap C([\zeta_-, 0] \times [0, m_{\text{in}}])$ due to the classical elliptic theory.

It follows from the proof of Proposition 3.1 in [20] that the following comparison principle holds.

Lemma 4.1 Assume that $q_+ \in L^\infty((\zeta_-, 0) \times (0, m_{\text{in}}))$ and $q_- \in L^\infty((\zeta_-, 0) \times (0, m_{\text{in}}))$ are a weak supersolution and a weak subsolution to the problem (3.31)–(3.34), (4.6), respectively. Then

$$q_+(\varphi, \psi) \geq q_-(\varphi, \psi), \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}).$$

4.2 Solutions to elliptic boundary problems

We now study solutions to the problem (3.31)–(3.34), (4.6) with $0 < c < c_*$. Due to Lemma 4.1, such a solution is unique. First we show the existence for sufficiently small c .

Lemma 4.2 There exists $\kappa_1 \in (0, 1]$ depending only on $\gamma, m, \lambda, \delta_1$ and δ_2 such that for any $-\kappa_1 \leq \zeta_- < 0$, the problem (3.31)–(3.34), (4.6) admits a unique solution with (4.7) and

$$\sup_{(0, m_{\text{in}})} q(\zeta_-, \cdot) \leq c_* - (\epsilon(\zeta_-) + \zeta_-^2)(-\zeta_-)^{\lambda/2+1} \quad (4.8)$$

if $0 < c \leq c_*/2$.

Proof. Set

$$\varepsilon_0 = \min \left\{ \frac{c_*}{12m_2^2}, \inf_{c_*/2 < s < 5c_*/6} \frac{A'(s)}{B'(s)} \right\}.$$

Choose $\kappa_1 \in (0, 1]$ such that

$$\kappa_1^2 \leq \frac{c_*}{6}, \quad \kappa_1 \leq \frac{c_*}{24}, \quad \kappa_1^\lambda \leq \frac{2\varepsilon_0 m_1}{\mu_2} \inf_{c_*/2 < s < 5c_*/6} B'(s).$$

Then,

$$q_+(\varphi, \psi) = \frac{2}{3}c_* - 2\varphi - \varphi^2 + \varepsilon_0\psi^2, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}]$$

is a supersolution to the problem (3.31)–(3.34), (4.6) if $0 < c \leq c_*/2$. In fact,

$$\frac{1}{2}c_* \leq q_+(\varphi, \psi) \leq \frac{5}{6}c_*, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}], \quad (4.9)$$

$$\begin{aligned} \frac{\partial^2 A(q_+)}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(q_+)}{\partial \psi^2}(\varphi, \psi) &\leq A'(q_+(\varphi, \psi)) \frac{\partial^2 q_+}{\partial \varphi^2}(\varphi, \psi) + B'(q_+(\varphi, \psi)) \frac{\partial^2 q_+}{\partial \psi^2}(\varphi, \psi) \\ &= -2A'(q_+(\varphi, \psi)) + 2\varepsilon_0 B'(q_+(\varphi, \psi)) \\ &\leq 0, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}), \end{aligned}$$

$$\frac{\partial A(q_+)}{\partial \varphi}(\zeta_-, \psi) = -2(1 + \zeta_-)A'(q_+(\zeta_-, \psi)) \leq 0, \quad \psi \in (0, m_{\text{in}}),$$

$$\frac{\partial B(q_+)}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (\zeta_-, 0),$$

$$\frac{\partial B(q_+)}{\partial \psi}(\varphi, m_{\text{in}}) = 2\varepsilon_0 m_{\text{in}} B'(q_+(\varphi, m_{\text{in}})) \geq \mu_2(-\varphi)^\lambda, \quad \varphi \in (\zeta_-, 0).$$

Additionally, it is not hard to verify that

$$q_-(\varphi, \psi) = A_-^{-1}(A(c) + \mu_2\varphi), \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}]$$

is a subsolution to the problem (3.31)–(3.34), (4.6). Then, one can get the existence and the uniqueness of the solution to the problem (3.31)–(3.34) by a standard argument in the classical elliptic theory. Finally, (4.8) follows from (4.9) and $(\epsilon(\zeta_-) + \zeta_-^2)(-\zeta_-)^{\lambda/2+1} \leq 2\kappa_1 \leq c_*/12$. \square

Below, we investigate properties of the solution to the problem (3.31)–(3.34), (4.6) with $0 < c < c_*$. As in the proof of Proposition 3.2 in [20], we introduce two functions. The first one is

$$E(s) = A(B^{-1}(s)), \quad s < 0.$$

Clearly, $E \in C^\infty((-\infty, 0))$ satisfies

$$\begin{aligned} E'(s) &= \left(1 - \frac{\gamma+1}{2}q^2\right) \left(1 - \frac{\gamma-1}{2}q^2\right)^{-2/(\gamma-1)-1} \Big|_{q=B^{-1}(s)} > 0, & s < 0, \\ E''(s) &= -(\gamma+1)q^4 \left(1 - \frac{\gamma-1}{2}q^2\right)^{-3/(\gamma-1)-2} \Big|_{q=B^{-1}(s)} < 0, & s < 0, \\ E'''(s) &= -(\gamma+1)q^4(4+3q^2) \left(1 - \frac{\gamma-1}{2}q^2\right)^{-4/(\gamma-1)-3} \Big|_{q=B^{-1}(s)} < 0, & s < 0. \end{aligned}$$

The other function is

$$G(s) = E'(E^{-1}(s)), \quad s < 0,$$

which satisfies $G \in C^\infty((-\infty, 0))$ and

$$\begin{aligned} G'(s) &= \frac{E''(E^{-1}(s))}{E'(E^{-1}(s))} = \frac{E''(t)}{E'(t)} \Big|_{t=E^{-1}(s)} < 0, & s < 0, \\ G''(s) &= \frac{E'''(t)E'(t) - (E''(t))^2}{(E'(t))^3} \Big|_{t=E^{-1}(s)} < -\frac{(E''(t))^2}{(E'(t))^3} \Big|_{t=E^{-1}(s)} < 0, & s < 0. \end{aligned}$$

Lemma 4.3 *Assume that $c_*/3 \leq c < c_*$ and q , satisfying (4.7) and (4.8), is the solution to the problem (3.31)–(3.34), (4.6). Then*

$$q(\varphi, \psi) \geq A_-^{-1}(A(c_*/3) + \mu_2\varphi), \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}], \quad (4.10)$$

$$\left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq \mu_3(C_2 + 1)(-\zeta_-)^{\lambda-1/2}(-\varphi)^{1/2}, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}] \quad (4.11)$$

and

$$\begin{aligned} \left| A(q(\varphi', \psi')) - A(q(\varphi'', \psi'')) \right| &\leq \mu_4(C_2 + 1)(|\varphi' - \varphi''|^{1/2} + |\psi' - \psi''|), \\ (\varphi', \psi'), (\varphi'', \psi'') &\in [\zeta_-, 0] \times [0, m_{\text{in}}], \end{aligned} \quad (4.12)$$

where $\mu_i = \mu_i(\gamma, m, \lambda, \delta_1, \delta_2, \underline{c})$ for $i = 3, 4$.

Proof. First, (4.10) follows from Lemma 4.1 and the fact that

$$q_-(\varphi, \psi) = A_-^{-1}(A(c_*/3) + \mu_2\varphi), \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}]$$

is a subsolution to the problem (3.31)–(3.34), (4.6). To prove (4.11), one can adapt the proof of Proposition 3.2 in [20] as follows. Set

$$z(\varphi, \psi) = \frac{\partial B(q)}{\partial \psi}(\varphi, \psi), \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}].$$

Then z solves the following problem

$$j_1(\varphi, \psi) \frac{\partial^2 z}{\partial \varphi^2} + \frac{\partial^2 z}{\partial \psi^2} + j_2(\varphi, \psi) \frac{\partial z}{\partial \varphi} + j_3(\varphi, \psi) \frac{\partial z}{\partial \psi} + j_4(\varphi, \psi) z = 0, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}), \quad (4.13)$$

$$\frac{\partial z}{\partial \varphi}(\zeta_-, \psi) + e(\psi) z(\zeta_-, \psi) = -\frac{B'(q(\zeta_-, \psi))}{A'(q(\zeta_-, \psi))} \left(\frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} \right)', \quad \psi \in (0, m_{\text{in}}), \quad (4.14)$$

$$z(\varphi, 0) = 0, \quad \varphi \in (\zeta_-, 0), \quad (4.15)$$

$$z(\varphi, m_{\text{in}}) = \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)}, \quad \varphi \in (\zeta_-, 0), \quad (4.16)$$

$$z(0, \psi) = 0, \quad \psi \in (0, m_{\text{in}}), \quad (4.17)$$

where for $(\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}})$,

$$j_1(\varphi, \psi) = G(A(q(\varphi, \psi))), \quad j_2(\varphi, \psi) = 2G'(A(q(\varphi, \psi))) \frac{\partial A(q(\varphi, \psi))}{\partial \varphi},$$

$$j_3(\varphi, \psi) = -G'(A(q(\varphi, \psi))) \frac{\partial B(q(\varphi, \psi))}{\partial \psi}, \quad j_4(\varphi, \psi) = G''(A(q(\varphi, \psi))) \left(\frac{\partial A(q(\varphi, \psi))}{\partial \varphi} \right)^2,$$

and

$$e(\psi) = -\frac{E''(q(\zeta_-, \psi))}{(E'(q(\zeta_-, \psi)))^2} \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} < 0, \quad \psi \in (0, m_{\text{in}}).$$

It follows from (4.10), (4.8) and (4.4) that there exists a positive number $\tilde{\mu}_3$ depending only on γ , m , λ , δ_1 , δ_2 and \underline{c} such that

$$\left| \frac{B'(q(\zeta_-, \psi))}{A'(q(\zeta_-, \psi))} \left(\frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} \right)' \right| \leq \frac{1}{2} \tilde{\mu}_3 (C_2 + 1) (-\zeta_-)^{\lambda-1}, \quad \psi \in (0, m_{\text{in}}).$$

Note

$$\begin{aligned} \frac{1}{4} j_1(\varphi, \psi) (-\varphi)^{-3/2} - j_4(\varphi, \psi) (-\varphi)^{1/2} &\geq \sqrt{-j_1(\varphi, \psi) j_4(\varphi, \psi)} (-\varphi)^{-1/2} \\ &\geq -\frac{1}{2} j_2(\varphi, \psi) (-\varphi)^{-1/2}, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}). \end{aligned}$$

One can verify that

$$z_{\pm}(\varphi, \psi) = \pm \tilde{\mu}_3 (C_2 + 1) (-\zeta_-)^{\lambda-1/2} (-\varphi)^{1/2}, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}]$$

are super and sub solutions to the problem (4.13)–(4.17). Then, (4.11) follows from the classical comparison principle. Finally, one can get (4.12) from (4.10) and (4.11) by a standard process as in the proof of Proposition 3.2 in [20]. \square

4.3 The weak solution to the degenerate elliptic boundary problem

We are now ready to solve the problem (3.31)–(3.35). First we determine \underline{c} by

$$\underline{c} = A_-^{-1}(A(c_*/3) - \mu_2) \in (0, c_*/3). \quad (4.18)$$

Proposition 4.1 *There exists $\kappa_2 \in (0, \kappa_1]$ depending only on $\gamma, m, \lambda, \delta_1, \delta_2, \epsilon(\cdot)$ and C_2 such that for any $-\kappa_2 \leq \zeta_- < 0$, the problem (3.31)–(3.35) admits a weak solution $q \in C^\infty((\zeta_-, 0) \times (0, m_{\text{in}})) \cap C^1([\zeta_-, 0] \times [0, m_{\text{in}}]) \cap C([\zeta_-, 0] \times [0, m_{\text{in}}])$ satisfying*

$$\left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq \mu_3(C_2 + 1)(-\zeta_-)^{\lambda-1/2}(-\varphi)^{1/2}, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}], \quad (4.19)$$

$$\begin{aligned} \left| A(q(\varphi', \psi')) - A(q(\varphi'', \psi'')) \right| &\leq \mu_4(C_2 + 1)(|\varphi' - \varphi''|^{1/2} + |\psi' - \psi''|), \\ (\varphi', \psi'), (\varphi'', \psi'') &\in [\zeta_-, 0] \times [0, m_{\text{in}}], \end{aligned} \quad (4.20)$$

$$c_* - \mu_6(-\varphi)^{\lambda/2+1} \leq q(\varphi, \psi) \leq c_* - \mu_5(-\varphi)^{\lambda/2+1}, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}], \quad (4.21)$$

where μ_5 and μ_6 ($\mu_5 \leq \mu_6$) depend only on $\gamma, m, \lambda, \delta_1$ and δ_2 .

Proof. Set

$$\mathcal{E} = \left\{ c_*/3 < c < c_* : \text{the problem (3.31)–(3.34), (4.6) admits a solution } q_c \text{ with (4.7) and (4.8)} \right\}.$$

According to Lemmas 4.1 and 4.2, $\mathcal{E} (\neq \emptyset)$ is an interval. We claim that $\mathcal{E} = (c_*/3, c_*)$ if $-\kappa_2 \leq \zeta_- < 0$, where $\kappa_2 \in (0, \kappa_1]$ will be determined.

Assume that $c \in \mathcal{E}$. It follows from Lemma 4.3 that q_c satisfies (4.11) and (4.12). Integrating (3.31) and (3.32) over $(0, m_{\text{in}})$ with respect to ψ and using (3.33) and (3.34), one gets that

$$\begin{aligned} \frac{d^2}{d\varphi^2} \int_0^{m_{\text{in}}} A(q_c(\varphi, \psi)) d\psi &= - \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)}, \quad \varphi \in (\zeta_-, 0), \\ \left(\frac{d}{d\varphi} \int_0^{m_{\text{in}}} A(q_c(\varphi, \psi)) d\psi \right) \Big|_{\varphi=\zeta_-} &= - \int_0^{m_{\text{in}}} \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} d\psi. \end{aligned} \quad (4.22)$$

Direction calculations yield

$$\int_{\zeta_-}^0 \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)} d\varphi = \int_{l_-}^0 \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Phi'_-(x) dx = \int_{l_-}^0 \Theta'_-(x) dx = -\Theta_-(-l_-)$$

and

$$\begin{aligned} \int_0^{m_{\text{in}}} \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Big|_{y=Y_{\text{in}}(\psi)} d\psi &= \int_0^{f_-(l_-)} \frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}(y) \rho(\mathcal{Q}_{\text{in}}^2(y))} \Psi'_{\text{in}}(y) dy \\ &= \int_0^{f_-(l_-)} \Theta'_{\text{in}}(y) dy = \Theta_{\text{in}}(f_-(l_-)). \end{aligned}$$

Therefore,

$$\left(\frac{d}{d\varphi} \int_0^{m_{\text{in}}} A(q_c(\varphi, \psi)) d\psi \right) \Big|_{\varphi=0} = 0. \quad (4.23)$$

It follows from (4.22), (4.23) and (4.6) that

$$\int_0^{m_{\text{in}}} A(q_c(\varphi, \psi)) d\psi = m_{\text{in}} A(c) - \int_{\varphi}^0 \int_{\tilde{\varphi}}^0 \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(s)} ds d\tilde{\varphi}, \quad \varphi \in [\zeta_-, 0],$$

which, together with (4.5), leads to

$$\int_0^{m_{\text{in}}} A(q_c(\varphi, \psi)) d\psi \leq m_{\text{in}} A(c) - \frac{\mu_1}{(\lambda+1)(\lambda+2)} (-\varphi)^{\lambda+2}, \quad \varphi \in [\zeta_-, 0].$$

Therefore, there exists a Lipschitz continuous curve $L : \psi = \omega(\varphi)$ ($\zeta_- \leq \varphi \leq 0$) in $[\zeta_-, 0] \times (0, m_{\text{in}})$ satisfying

$$A(q_c(\varphi, \omega(\varphi))) \leq -\frac{\mu_1}{m_{\text{in}}(\lambda+1)(\lambda+2)} (-\varphi)^{\lambda+2}, \quad \varphi \in [\zeta_-, 0].$$

Thus, there exists $M_0 = M_0(\gamma, m, \lambda, \delta_1, \delta_2)$ such that

$$q_c(\varphi, \omega(\varphi)) \leq c_* - M_0(-\varphi)^{\lambda/2+1}, \quad \varphi \in [\zeta_-, 0], \quad (4.24)$$

which, together with (4.11), leads to

$$q_c(\zeta_-, \psi) \leq c_* - M_0(-\zeta_-)^{\lambda/2+1} + \mu_3(C_2 + 1)m_2(-\zeta_-)^{\lambda}, \quad \psi \in [0, m_{\text{in}}]. \quad (4.25)$$

Decompose $(\zeta_-, 0) \times (0, m_{\text{in}})$ into Ω_1 and Ω_2 by the curve L , where

$$\Omega_1 = \{(\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}) : \psi < \omega(\varphi)\}, \quad \Omega_2 = \{(\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}) : \psi > \omega(\varphi)\}.$$

Then, q_c solves the following two problems

$$\begin{aligned} \frac{\partial^2 A(q_c)}{\partial \varphi^2} + \frac{\partial^2 B(q_c)}{\partial \psi^2} &= 0, & (\varphi, \psi) \in \Omega_1, \\ q_c(\zeta_-, \psi) &\leq c_* - M_0(-\zeta_-)^{\lambda/2+1} + \mu_3(C_2 + 1)m_2(-\zeta_-)^{\lambda}, & \psi \in (0, \omega(\zeta_-)), \\ \frac{\partial q_c}{\partial \psi}(\varphi, 0) &= 0, & \varphi \in (\zeta_-, 0), \\ q_c(\varphi, \omega(\varphi)) &\leq c_* - M_0(-\varphi)^{\lambda/2+1}, & \varphi \in (\zeta_-, 0), \\ q_c(0, \psi) &= c, & \psi \in (0, \omega(0)) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 A(q_c)}{\partial \varphi^2} + \frac{\partial^2 B(q_c)}{\partial \psi^2} &= 0, & (\varphi, \psi) \in \Omega_2, \\ q_c(\zeta_-, \psi) &\leq c_* - M_0(-\zeta_-)^{\lambda/2+1} + \mu_3(C_2 + 1)m_2(-\zeta_-)^{\lambda}, & \psi \in (\omega(\zeta_-), m_{\text{in}}), \\ q_c(\varphi, \omega(\varphi)) &\leq c_* - M_0(-\varphi)^{\lambda/2+1}, & \varphi \in (\zeta_-, 0), \end{aligned}$$

$$\begin{aligned}\frac{\partial B(q_c)}{\partial \psi}(\varphi, m_{\text{in}}) &= \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-(x)} \Big|_{x=X_-(\varphi)}, & \varphi \in (\zeta_-, 0), \\ q_c(0, \psi) &= c, & \psi \in (\omega(0), m_{\text{in}}).\end{aligned}$$

Set

$$\bar{q}_1(\varphi, \psi) = c_* - \mu_5(-\varphi)^{\lambda/2+1}, \quad (\varphi, \psi) \in \bar{\Omega}_1$$

and

$$\bar{q}_2(\varphi, \psi) = c_* - \mu_5(-\varphi)^{\lambda/2+1}(1 + m_2 - \psi), \quad (\varphi, \psi) \in \bar{\Omega}_2$$

with $\mu_5 = \frac{M_0}{2(1+m_2)}$. Note that \bar{q}_1 and \bar{q}_2 satisfy

$$\frac{\partial^2 A(\bar{q}_k)}{\partial \varphi^2}(\varphi, \psi) + \frac{\partial^2 B(\bar{q}_k)}{\partial \psi^2}(\varphi, \psi) \leq 0, \quad (\varphi, \psi) \in \Omega_k, \quad k = 1, 2.$$

Choose a number $\tilde{\kappa}_2 \in (0, \kappa_1]$ depending only on $\gamma, m, \lambda, \delta_1, \delta_2, \epsilon(\cdot)$ and C_2 , such that

$$\mu_3(C_2 + 1)m_2\tilde{\kappa}_2^{\lambda/2-1} \leq \frac{M_0}{2}, \quad \sup_{0 < s < \tilde{\kappa}_2} \epsilon(-s) + \tilde{\kappa}_2^2 \leq \frac{M_0}{4}, \quad \mu_2\tilde{\kappa}_2^{\lambda/2-1} \leq \mu_5 \inf_{\underline{c} < s < c_*} B'(s). \quad (4.26)$$

Owing to (4.26) and (4.5), one can get from the classical comparison principle that

$$q_c(\varphi, \psi) \leq \bar{q}_k(\varphi, \psi), \quad (\varphi, \psi) \in \bar{\Omega}_k, \quad k = 1, 2.$$

Therefore,

$$q_c(\varphi, \psi) \leq c_* - \mu_5(-\varphi)^{\lambda/2+1}, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}]. \quad (4.27)$$

Owing to (4.12) and (4.6), one also gets that

$$A(q_c(\varphi, \psi)) \leq A(c) + \mu_4(C_2 + 1)(-\varphi)^{1/2}, \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}]. \quad (4.28)$$

Moreover, it follows from (4.25) and (4.26) that

$$q_c(\zeta_-, \psi) \leq c_* - 2(\epsilon(\zeta_-) + \zeta_-^2)(-\zeta_-)^{\lambda/2+1}, \quad 0 \leq \psi \leq m_{\text{in}}. \quad (4.29)$$

Set $c_0 = \sup \mathcal{E}$. We show that $c_0 = c_*$ by contradiction. Otherwise, $c_0 < c_*$. Then, for any $c_*/3 < c < c_0$, $c \in \mathcal{E}$ and q_c satisfies (4.27)–(4.29), where q_c is the solution to the problem (3.31)–(3.34), (4.6). Set

$$q_{c_0}(\varphi, \psi) = \lim_{c \rightarrow c_0^-} q_c(\varphi, \psi), \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}].$$

Then, it follows from (4.27)–(4.29) that

$$\sup_{(\zeta_-, 0) \times (0, m_{\text{in}})} q_{c_0} < c_*, \quad \sup_{(0, m_{\text{in}})} q_{c_0}(\zeta_-, \cdot) \leq c_* - 2(\epsilon(\zeta_-) + \zeta_-^2)(-\zeta_-)^{\lambda/2+1}, \quad (4.30)$$

and it is not hard to show that q_{c_0} is the solution to the problem (3.31)–(3.34), (4.6) with $c = c_0$. Hence $c_0 \in \mathcal{E}$. It follows from (4.30) and the stability theory of uniformly elliptic problems that the problem (3.31)–(3.34), (4.6) admits a unique solution with (4.7) and (4.8) if $c_0 < c < c_0 + \varepsilon$ with a sufficiently small positive number ε . This contradicts that $c_0 = \sup \mathcal{E} < c_*$. Therefore, $\mathcal{E} = (c_*/3, c_*)$.

For any $c_*/3 < c < c_*$, let q_c be the solution to the problem (3.31)–(3.34), (4.6) with (4.7) and (4.8). Then, q_c satisfies (4.11), (4.12), (4.27) and (4.29). Set

$$q(\varphi, \psi) = \lim_{c \rightarrow c_*^-} q_c(\varphi, \psi), \quad (\varphi, \psi) \in [\zeta_-, 0] \times [0, m_{\text{in}}].$$

One can show that q is the weak solution to the problem (3.31)–(3.35) satisfying (4.19), (4.20) and the second inequality in (4.21). Moreover, $q \in C^\infty((\zeta_-, 0) \times (0, m_{\text{in}})) \cap C^1([\zeta_-, 0] \times [0, m_{\text{in}}]) \cap C([\zeta_-, 0] \times [0, m_{\text{in}}])$ thanks to the Schauder theory on uniformly elliptic equations.

It remains to verify the first inequality in (4.21). Similar to the proof of (4.24), there exists a Lipschitz continuous curve $\tilde{L} : \psi = \tilde{\omega}(\varphi)$ ($\zeta_- \leq \varphi \leq 0$) in $[\zeta_-, 0] \times (0, m_{\text{in}})$ satisfying

$$q(\varphi, \tilde{\omega}(\varphi)) \geq c_* - M_1(-\varphi)^{\lambda/2+1}, \quad \varphi \in [\zeta_-, 0]$$

with $M_1 = M_1(\gamma, m, \lambda, \delta_1, \delta_2)$. Decompose $(\zeta_-, 0) \times (0, m_{\text{in}})$ into $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ by the curve \tilde{L} , where

$$\tilde{\Omega}_1 = \{(\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}) : \psi < \tilde{\omega}(\varphi)\}, \quad \tilde{\Omega}_2 = \{(\varphi, \psi) \in (\zeta_-, 0) \times (0, m_{\text{in}}) : \psi > \tilde{\omega}(\varphi)\}.$$

Prescribe a Dirichlet condition for q on \tilde{L} . Then the problem (3.31)–(3.35) can be regarded as two problems on $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, respectively. Set

$$\begin{aligned} \underline{q}_1(\varphi, \psi) &= c_* - M_2(-\varphi)^{\lambda/2+1}(1 + m_{\text{in}}^2 - \psi^2), \quad (\varphi, \psi) \in \tilde{\Omega}_1, \\ \underline{q}_2(\varphi, \psi) &= c_* - M_2(-\varphi)^{\lambda/2+1}(1 + m_{\text{in}}^2 - \psi^2 + 2m_{\text{in}}\psi), \quad (\varphi, \psi) \in \tilde{\Omega}_2. \end{aligned}$$

Owing to $\lambda > 2$, (4.3) and (4.5), there exist sufficiently small $\kappa_2 \in (0, \tilde{\kappa}_2]$ and sufficiently large M_2 such that \underline{q}_k is a subsolution to the problem of q on $\tilde{\Omega}_k$ for $k = 1, 2$, where M_2 depends on $\gamma, m, \lambda, \delta_1$ and δ_2 , while κ_2 also on $\epsilon(\cdot)$ and C_2 . Then, the first inequality in (4.21) follows from a comparison principle. Here, the comparison principle is not the classical one since (3.31) is degenerate at the sonic state and it can be proved in a similar way as for Proposition 3.1 in [20]. \square

4.4 Existence of subsonic-sonic flows

We now choose

$$C_1 = \mu_5 \underline{c}^{\lambda/2+1}, \quad C_2 = \mu_6 \left(c_* \sqrt{\delta_2^2 + 1} \right)^{\lambda/2+1}. \quad (4.31)$$

Assume that $f_- \in C^3([l_-, 0])$ satisfies (3.22) with $-\delta_0 \leq l_- < 0$, where

$$\delta_0 = \frac{1}{c_* \sqrt{\delta_2^2 + 1}} \min \left\{ \kappa_2, \left(\frac{1}{\mu_3(C_2 + 1)c_* \rho(c_*^2) \sqrt{\delta_2^2 + 1}} \right)^{1/\lambda}, \left(\frac{c_*}{2\mu_6} \right)^{2/(\lambda+2)} \right\}, \quad (4.32)$$

which depends only on $\gamma, m, \lambda, \delta_1, \delta_2$ and $\epsilon(\cdot)$. Assume that $g \in C^3([0, f_-(l_-)])$ satisfies (3.23) and (3.24). Choose

$$C_0 = \mu_4(C_2 + 1) \left(c_* \sqrt{\delta_2^2 + 1} \right)^{1/2} \sup \left\{ \frac{1}{A'(s)} : \underline{c} < s < c_* - \mu_5 \left(\frac{-cl_-}{2} \right)^{\lambda/2+1} \right\}. \quad (4.33)$$

Set

$$\mathcal{S} = \left\{ (\mathcal{Q}_{\text{in}}, \mathcal{Q}_-) \in C^{0,1}([0, f_-(l_-)]) \times C([l_-, 0]) \cap C^{1/2}([l_-, l_-/2]) : \right.$$

\mathcal{Q}_{in} satisfies (4.1) with (4.31), while \mathcal{Q}_{-} satisfies (4.2) with (4.18) and (4.33) $\Big\}$

with the norm

$$\|(\mathcal{Q}_{\text{in}}, \mathcal{Q}_{-})\|_{\mathcal{S}} = \max \left\{ \|\mathcal{Q}_{\text{in}}\|_{L^{\infty}((0, f_{-}(l_{-})))}, \|\mathcal{Q}_{-}\|_{L^{\infty}((l_{-}, 0))} \right\}, \quad (\mathcal{Q}_{\text{in}}, \mathcal{Q}_{-}) \in \mathcal{S}.$$

For given $(\mathcal{Q}_{\text{in}}, \mathcal{Q}_{-}) \in \mathcal{S}$, it follows from Lemma 4.1 and Proposition 4.1 that the problem (3.31)–(3.35) admits a unique weak solution $q \in C^{\infty}((\zeta_{-}, 0) \times (0, m_{\text{in}})) \cap C^1([\zeta_{-}, 0] \times [0, m_{\text{in}}]) \cap C([\zeta_{-}, 0] \times [0, m_{\text{in}}])$ satisfying the estimates (4.19)–(4.21). Set

$$\hat{\mathcal{Q}}_{\text{in}}(y) = q(\zeta_{-}, \Psi_{\text{in}}(y)), \quad 0 \leq y \leq f_{-}(l_{-})$$

and

$$\hat{\mathcal{Q}}_{-}(x) = q(\Phi_{-}(x), m_{\text{in}}), \quad l_{-} \leq x \leq 0.$$

It follows from (4.18)–(4.21) and (4.31)–(4.33) that $\hat{\mathcal{Q}}_{\text{in}} \in C^{0,1}([0, f_{-}(l_{-})])$ satisfies (4.1), and $\hat{\mathcal{Q}}_{-} \in C([l_{-}, 0]) \cap C^{1/2}([l_{-}, l_{-}/2])$ satisfies (4.2). Therefore, we can define a mapping J from \mathcal{S} to itself as follows

$$J((\mathcal{Q}_{\text{in}}, \mathcal{Q}_{-})) = (\hat{\mathcal{Q}}_{\text{in}}, \hat{\mathcal{Q}}_{-}), \quad (\mathcal{Q}_{\text{in}}, \mathcal{Q}_{-}) \in \mathcal{S}. \quad (4.34)$$

Theorem 4.1 *Assume that $f_{-} \in C^3([l_{-}, 0])$ satisfies (3.22), and $g \in C^3([0, f_{-}(l_{-})])$ satisfies (3.23) and (3.24). If $-\delta_0 \leq l_{-} < 0$, then, the problem (3.31)–(3.37) admits a weak solution $q \in C^{\infty}((\zeta_{-}, 0) \times (0, m_{\text{in}})) \cap C^1([\zeta_{-}, 0] \times [0, m_{\text{in}}]) \cap C([\zeta_{-}, 0] \times [0, m_{\text{in}}])$ with the following estimates*

$$\left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq \mu_3(C_2 + 1)(-\zeta_{-})^{\lambda-1/2}(-\varphi)^{1/2}, \quad (\varphi, \psi) \in (\zeta_{-}, 0) \times (0, m_{\text{in}}),$$

$$\begin{aligned} \left| A(q(\varphi', \psi')) - A(q(\varphi'', \psi'')) \right| &\leq \mu_4(C_2 + 1)(|\varphi' - \varphi''|^{1/2} + |\psi' - \psi''|), \\ (\varphi', \psi'), (\varphi'', \psi'') &\in (\zeta_{-}, 0) \times (0, m_{\text{in}}) \end{aligned}$$

and

$$c_{*} - \mu_6(-\varphi)^{\lambda/2+1} \leq q(\varphi, \psi) \leq c_{*} - \mu_5(-\varphi)^{\lambda/2+1}, \quad (\varphi, \psi) \in (\zeta_{-}, 0) \times (0, m_{\text{in}}),$$

where δ_0 and C_2 are defined in (4.32) and (4.31), respectively, while μ_i ($i = 3, 4, 5, 6$) are given in Lemma 4.3 and Proposition 4.1. Furthermore,

$$m_{\text{in}} = m. \quad (4.35)$$

Proof. As mentioned above, the mapping J defined by (4.34) is from \mathcal{S} to itself. It follows from (4.19)–(4.21) that J is compact. Therefore, the first part of the theorem follows from Proposition 4.1 and the Schauder fixed point theorem provided that J is also continuous.

Now let us show that J is continuous. Assume that $\{(\mathcal{Q}_{\text{in}}^{(n)}, \mathcal{Q}_{-}^{(n)})\}_{n=0}^{\infty} \subset \mathcal{S}$ satisfies

$$\lim_{n \rightarrow \infty} \|\mathcal{Q}_{\text{in}}^{(n)} - \mathcal{Q}_{\text{in}}^{(0)}\|_{L^{\infty}((0, f_{-}(l_{-})))} = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{Q}_{-}^{(n)} - \mathcal{Q}_{-}^{(0)}\|_{L^{\infty}((l_{-}, 0))} = 0. \quad (4.36)$$

Let $m_{\text{in}}^{(n)}$, $\zeta_{-}^{(n)}$, $\Psi_{\text{in}}^{(n)}$, $Y_{\text{in}}^{(n)}$, $\Phi_{-}^{(n)}$ and $X_{-}^{(n)}$ be defined by (3.25)–(3.30) with $\mathcal{Q}_{\text{in}} = \mathcal{Q}_{\text{in}}^{(n)}$ and $\mathcal{Q}_{-} = \mathcal{Q}_{-}^{(n)}$ for $n = 0, 1, 2, \dots$. Then, (4.36) yields

$$\lim_{n \rightarrow \infty} m_{\text{in}}^{(n)} = m_{\text{in}}^{(0)}, \quad \lim_{n \rightarrow \infty} \zeta_{-}^{(n)} = \zeta_{-}^{(0)}, \quad (4.37)$$

$$\lim_{n \rightarrow \infty} \|\Psi_{\text{in}}^{(n)} - \Psi_{\text{in}}^{(0)}\|_{0,1;(0,f_{-}(l_{-}))} = 0, \quad Y_{\text{in}}^{(n)} \longrightarrow Y_{\text{in}}^{(0)} \text{ uniformly in } (0, m_{\text{in}}^{(0)}), \quad (4.38)$$

$$\lim_{n \rightarrow \infty} \|\Phi_{-}^{(n)} - \Phi_{-}^{(0)}\|_{0,1;(l_{-},0)} = 0, \quad X_{-}^{(n)} \longrightarrow X_{-}^{(0)} \text{ uniformly in } (\zeta_{-}^{(0)}, 0). \quad (4.39)$$

It follows from the definition of J that

$$J((\mathcal{Q}_{\text{in}}^{(n)}, \mathcal{Q}_{-}^{(n)})) = (q^{(n)}(\zeta_{-}^{(n)}, \Psi_{\text{in}}^{(n)}(\cdot)), q^{(n)}(\Phi_{-}^{(n)}(\cdot), m_{\text{in}}^{(n)})), \quad n = 0, 1, 2, \dots,$$

where $q^{(n)} \in C^\infty((\zeta_{-}^{(n)}, 0) \times (0, m_{\text{in}}^{(n)})) \cap C^1([\zeta_{-}^{(n)}, 0] \times [0, m_{\text{in}}^{(n)}]) \cap C([\zeta_{-}^{(n)}, 0] \times [0, m_{\text{in}}^{(n)}])$ is the unique weak solution to the following problem

$$\frac{\partial^2 A(q^{(n)})}{\partial \varphi^2} + \frac{\partial^2 B(q^{(n)})}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (\zeta_{-}^{(n)}, 0) \times (0, m_{\text{in}}^{(n)}), \quad (4.40)$$

$$\frac{\partial A(q^{(n)})}{\partial \varphi}(\zeta_{-}^{(n)}, \psi) = -\frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}^{(n)}(y) \rho((\mathcal{Q}_{\text{in}}^{(n)}(y))^2)} \Big|_{y=Y_{\text{in}}^{(n)}(\psi)}, \quad \psi \in (0, m_{\text{in}}^{(n)}), \quad (4.41)$$

$$\frac{\partial q^{(n)}}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (\zeta_{-}^{(n)}, 0), \quad (4.42)$$

$$\frac{\partial B(q^{(n)})}{\partial \psi}(\varphi, m_{\text{in}}^{(n)}) = \frac{\Theta'_{-}(x) \cos \Theta_{-}(x)}{\mathcal{Q}_{-}^{(n)}(x)} \Big|_{x=X_{-}^{(n)}(\varphi)}, \quad \varphi \in (\zeta_{-}^{(n)}, 0), \quad (4.43)$$

$$q^{(n)}(0, \psi) = c_*, \quad \psi \in (0, m_{\text{in}}^{(n)}) \quad (4.44)$$

for $n = 0, 1, 2, \dots$. Set

$$\hat{q}^{(n)}(\hat{\varphi}, \hat{\psi}) = q^{(n)}(\zeta_{-}^{(n)} \hat{\varphi}, m_{\text{in}}^{(n)} \hat{\psi}), \quad (\hat{\varphi}, \hat{\psi}) \in (0, 1) \times (0, 1), \quad n = 0, 1, 2, \dots$$

It remains to verify that

$$\lim_{n \rightarrow \infty} (\|\hat{q}^{(n)}(1, \cdot) - \hat{q}^{(0)}(1, \cdot)\|_{L^\infty((0,1))} + \|\hat{q}^{(n)}(\cdot, 1) - \hat{q}^{(0)}(\cdot, 1)\|_{L^\infty((0,1))}) = 0, \quad (4.45)$$

which, together with (4.37)–(4.39), implies

$$\lim_{n \rightarrow \infty} \|J((\mathcal{Q}_{\text{in}}^{(n)}, \mathcal{Q}_{-}^{(n)})) - J((\mathcal{Q}_{\text{in}}^{(0)}, \mathcal{Q}_{-}^{(0)}))\|_{\mathcal{J}} = 0.$$

We prove (4.45) by contradiction. Otherwise, there exist a positive number ε_0 and a subsequence of $\{q^{(n)}\}_{n=1}^\infty$, denoted by itself for convenience, such that for each $n = 1, 2, \dots$,

$$\|\hat{q}^{(n)}(1, \cdot) - \hat{q}^{(0)}(1, \cdot)\|_{L^\infty((0,1))} + \|\hat{q}^{(n)}(\cdot, 1) - \hat{q}^{(0)}(\cdot, 1)\|_{L^\infty((0,1))} \geq \varepsilon_0. \quad (4.46)$$

Since $q^{(n)}$ satisfies (4.20) and (4.21) with $m_{\text{in}} = m_{\text{in}}^{(n)}$ and $\zeta_{-} = \zeta_{-}^{(n)}$ for each $n = 1, 2, \dots$, there exists a subsequence of $\{q^{(n)}\}_{n=1}^\infty$, denoted by itself again for convenience, such that

$$\lim_{n \rightarrow \infty} \|\hat{q}^{(n)} - \hat{q}^*\|_{L^\infty((0,1) \times (0,1))} = 0, \quad (4.47)$$

where

$$\hat{q}^*(\hat{\varphi}, \hat{\psi}) = q^*(\zeta_-^{(0)} \hat{\varphi}, m_{\text{in}}^{(0)} \hat{\psi}), \quad (\hat{\varphi}, \hat{\psi}) \in (0, 1) \times (0, 1)$$

and $q^* \in L^\infty((\zeta_-^{(0)}, 0) \times (0, m_{\text{in}}^{(0)}))$ satisfies (4.20) and (4.21) with $m_{\text{in}} = m_{\text{in}}^{(0)}$ and $\zeta_- = \zeta_-^{(0)}$. Letting $n \rightarrow \infty$ in (4.40)–(4.44) and using (4.37)–(4.39) and (4.47), one can get that q^* solves the following problem

$$\begin{aligned} \frac{\partial^2 A(q^*)}{\partial \varphi^2} + \frac{\partial^2 B(q^*)}{\partial \psi^2} &= 0, & (\varphi, \psi) &\in (\zeta_-^{(0)}, 0) \times (0, m_{\text{in}}^{(0)}), \\ \frac{\partial A(q^*)}{\partial \varphi}(\zeta_-^{(0)}, \psi) &= -\frac{\Theta'_{\text{in}}(y) \cos \Theta_{\text{in}}(y)}{\mathcal{Q}_{\text{in}}^{(0)}(y) \rho((\mathcal{Q}_{\text{in}}^{(0)}(y))^2)} \Big|_{y=Y_{\text{in}}^{(0)}(\psi)}, & \psi &\in (0, m_{\text{in}}^{(0)}), \\ \frac{\partial q^*}{\partial \psi}(\varphi, 0) &= 0, & \varphi &\in (\zeta_-^{(0)}, 0), \\ \frac{\partial B(q^*)}{\partial \psi}(\varphi, m_{\text{in}}^{(0)}) &= \frac{\Theta'_-(x) \cos \Theta_-(x)}{\mathcal{Q}_-^{(0)}(x)} \Big|_{x=X_-^{(0)}(\varphi)}, & \varphi &\in (\zeta_-^{(0)}, 0), \\ q^*(0, \psi) &= c_*, & \psi &\in (0, m_{\text{in}}^{(0)}). \end{aligned}$$

It follows from Lemma 4.1 that

$$q^*(\varphi, \psi) = q^{(0)}(\varphi, \psi), \quad (\varphi, \psi) \in (\zeta_-^{(0)}, 0) \times (0, m_{\text{in}}^{(0)}),$$

which contradicts (4.46) and (4.47). Hence (4.45) holds. Finally, it follows from (3.14) that the sonic curve of the flow lies at the throat of the nozzle and the velocity vector is along the normal direction at the sonic curve in the physical plane. Thus (4.35) holds. \square

4.5 Uniqueness of subsonic-sonic flows

We show the uniqueness of the subsonic-sonic flow in the physical plane. Note that the boundary condition (3.14) means that the velocity vector of the flow is along the normal direction at the sonic curve. Thus, for the problem (3.10)–(3.15), the solution satisfies both a Dirichlet and a Neumann boundary conditions at the sonic curve although there is a free parameter at the inlet. Hence the uniqueness theorem follows easily.

Theorem 4.2 *There exists at most one solution $\varphi \in C^{1,1}(\overline{\Omega_-})$ to the problem (3.10)–(3.15).*

Proof. Let $\varphi_1, \varphi_2 \in C^{1,1}(\overline{\Omega_-})$ be two solutions to the problem (3.10)–(3.15). Assume that

$$\varphi_k(g(y), y) = C_{\text{in},k}, \quad 0 < y < f_-(l_-), \quad k = 1, 2.$$

For $k = 1, 2$, multiplying the equation with respect to φ_k by $(\varphi_1 - \varphi_2) - (C_{\text{in},1} - C_{\text{in},2})$ and then integrating over Ω_- by parts, one can get that

$$\int_{\Omega_-} \rho(|\nabla \varphi_k(x, y)|^2) \nabla \varphi_k(x, y) \cdot \nabla (\varphi_1(x, y) - \varphi_2(x, y)) dx dy + (C_{\text{in},1} - C_{\text{in},2}) \rho(c_*^2) c_* f_-(0) = 0.$$

Hence

$$\int_{\Omega_-} (\rho(|\nabla \varphi_1(x, y)|^2) \nabla \varphi_1(x, y) - \rho(|\nabla \varphi_2(x, y)|^2) \nabla \varphi_2(x, y)) \cdot (\nabla \varphi_1(x, y) - \nabla \varphi_2(x, y)) dx dy = 0,$$

which, together with (3.14) and (3.15), leads to

$$\varphi_1(x, y) = \varphi_2(x, y), \quad (x, y) \in \Omega_-.$$

□

4.6 Regularity of the subsonic-sonic flow

The subsonic-sonic flow in Theorem 4.1 is only continuous near the sonic state. We investigate its regularity in this subsection.

Theorem 4.3 *Let q be the weak solution to the problem (3.31)–(3.37) under the assumptions of Theorem 4.1.*

(i) *If f_- satisfies (3.65) additionally, then $q \in C^1([\zeta_-, 0] \times [0, m])$ satisfies*

$$\left| \frac{\partial q}{\partial \varphi}(\varphi, \psi) \right| \leq \mu_7(-\varphi)^{\lambda/4+1/2}, \quad \left| \frac{\partial q}{\partial \psi}(\varphi, \psi) \right| \leq \mu_7(-\varphi)^{\lambda/2+1}, \quad (\varphi, \psi) \in (\zeta_-, 0) \times (0, m) \quad (4.48)$$

with $\mu_7 = \mu_7(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3)$.

(ii) *If $f_- \in C^4([l_-, 0])$ satisfies (3.65) and (3.67) additionally, then $q \in C^{1,1}((\zeta_-, 0] \times [0, m])$ satisfies*

$$\left| \frac{\partial^2 q}{\partial \varphi^2}(\varphi, \psi) \right| \leq \mu_8, \quad \left| \frac{\partial^2 q}{\partial \varphi \partial \psi}(\varphi, \psi) \right| \leq \mu_8(-\varphi)^{\lambda/4+1/2}, \quad \left| \frac{\partial^2 q}{\partial \psi^2}(\varphi, \psi) \right| \leq \mu_8(-\varphi)^{\lambda/2+1},$$

$$(\varphi, \psi) \in (\zeta_-/2, 0) \times (0, m) \quad (4.49)$$

with $\mu_8 = \mu_8(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \delta_4)$.

(iii) *If $g \in C^{3,\alpha}([0, f_-(l_-)])$ and $f_- \in C^{3,\alpha}([l_-, 0])$ for a number $\alpha \in (0, 1)$ additionally, then $q \in C^{2,\alpha}([\zeta_-, 0] \times [0, m])$.*

Proof. Thanks to the Schauder theory on uniformly elliptic equations, one can get that (4.48) holds in $(\zeta_-, \zeta_-/2) \times (0, m)$ in the case (i), $q \in C^{1,1}((\zeta_-, 0) \times [0, m])$ in the case (ii) and $q \in C^{2,\alpha}([\zeta_-, 0] \times [0, m])$ in the case (iii). So it suffices to verify (4.48) and (4.49) in $(\zeta_-/2, 0) \times (0, m)$.

For any $0 < \varepsilon < -\zeta_-/2 \leq 1/2$, $q \in C^2([-2\varepsilon, -\varepsilon/2] \times (0, m)) \cap C^1([-2\varepsilon, -\varepsilon/2] \times [0, m])$ satisfies

$$\begin{aligned} \frac{\partial}{\partial \varphi} \left(a(\varphi, \psi) \frac{\partial q}{\partial \varphi} \right) + \frac{\partial}{\partial \psi} \left(h(\varphi, \psi) \frac{\partial q}{\partial \psi} \right) &= 0, & (\varphi, \psi) &\in (-2\varepsilon, -\varepsilon/2) \times (0, m), \\ \frac{\partial q}{\partial \psi}(\varphi, 0) &= 0, & \varphi &\in (-2\varepsilon, -\varepsilon/2), \\ \frac{\partial q}{\partial \psi}(\varphi, m) &= e(\varphi), & \varphi &\in (-2\varepsilon, -\varepsilon/2), \end{aligned}$$

where

$$a(\varphi, \psi) = A'(q(\varphi, \psi)), \quad h(\varphi, \psi) = B'(q(\varphi, \psi)), \quad (\varphi, \psi) \in [-2\varepsilon, -\varepsilon/2] \times [0, m]$$

and

$$e(\varphi) = \frac{\Theta'_-(X_-(\varphi)) \cos \Theta_-(X_-(\varphi))}{B'(q(\varphi, m))q(\varphi, m)}, \quad \varphi \in [-2\varepsilon, -\varepsilon/2].$$

Shifting and rescaling the potential as

$$\tilde{\varphi} = \varepsilon^{-\lambda/4-1/2}(\varphi + \varepsilon), \quad \varphi \in [-2\varepsilon, -\varepsilon/2]$$

and setting

$$\tilde{q}(\tilde{\varphi}, \psi) = q(-\varepsilon + \varepsilon^{\lambda/4+1/2}\tilde{\varphi}, \psi) - c_*, \quad (\tilde{\varphi}, \psi) \in [-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2] \times [0, m],$$

one can check that $\tilde{q} \in C^2([-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2] \times (0, m)) \cap C^1([-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2] \times [0, m])$ solves

$$\begin{aligned} \frac{\partial}{\partial \tilde{\varphi}} \left(\varepsilon^{-\lambda/2-1} \tilde{a}(\tilde{\varphi}, \psi) \frac{\partial \tilde{q}}{\partial \tilde{\varphi}} \right) + \frac{\partial}{\partial \psi} \left(\tilde{h}(\tilde{\varphi}, \psi) \frac{\partial \tilde{q}}{\partial \psi} \right) &= 0, & (\tilde{\varphi}, \psi) &\in (-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2) \times (0, m), \\ \frac{\partial \tilde{q}}{\partial \psi}(\tilde{\varphi}, 0) &= 0, & \tilde{\varphi} &\in (-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2), \\ \frac{\partial \tilde{q}}{\partial \psi}(\tilde{\varphi}, m) &= \tilde{e}(\tilde{\varphi}), & \tilde{\varphi} &\in (-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2), \end{aligned}$$

where \tilde{a} and \tilde{h} are defined on $[-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2] \times [0, m]$ by

$$\tilde{a}(\tilde{\varphi}, \psi) = a(-\varepsilon + \varepsilon^{\lambda/4+1/2}\tilde{\varphi}, \psi), \quad \tilde{h}(\tilde{\varphi}, \psi) = h(-\varepsilon + \varepsilon^{\lambda/4+1/2}\tilde{\varphi}, \psi),$$

and

$$\tilde{e}(\tilde{\varphi}) = e(-\varepsilon + \varepsilon^{\lambda/4+1/2}\tilde{\varphi}), \quad \tilde{\varphi} \in [-\varepsilon^{-\lambda/4+1/2}, \varepsilon^{-\lambda/4+1/2}/2].$$

It follows from the Hölder continuity estimates for uniformly elliptic equations that there exists a number $\beta \in (0, 1)$ such that

$$\|\tilde{q}\|_{\beta; (-1/4, 1/4) \times (0, m)} \leq M_1 \|\tilde{q}\|_{L^\infty((-1/2, 1/2) \times (0, m))} \leq M_2 \varepsilon^{\lambda/2+1}, \quad (4.50)$$

where $M_i = M_i(\gamma, m, \lambda, \delta_1, \delta_2)$ for $i = 1, 2$. Then, (4.50) and (3.65) yield

$$\varepsilon^{-\lambda/2-1} \|\tilde{a}\|_{\beta; (-1/4, 1/4) \times (0, m)} + \|\tilde{h}\|_{\beta; (-1/4, 1/4) \times (0, m)} \leq M_3, \quad \|\tilde{e}\|_{\beta; (-1/4, 1/4)} \leq M_3 \varepsilon^{\lambda/2+1}$$

with $M_3 = M_3(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3)$. So one gets from the Schauder estimates on uniformly elliptic equations that

$$\|\tilde{q}\|_{1, \beta; (-1/8, 1/8) \times (0, m)} \leq M_4 (\|\tilde{q}\|_{L^\infty((-1/4, 1/4) \times (0, m))} + \|\tilde{e}\|_{\beta; (-1/4, 1/4)}) \leq M_5 \varepsilon^{\lambda/2+1}, \quad (4.51)$$

where $M_i = M_i(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3)$ for $i = 4, 5$. It follows from (4.51) that

$$\left| \frac{\partial q}{\partial \varphi}(-\varepsilon, \psi) \right| \leq M_6 \varepsilon^{\lambda/4+1/2}, \quad \left| \frac{\partial q}{\partial \psi}(-\varepsilon, \psi) \right| \leq M_6 \varepsilon^{\lambda/2+1}, \quad \psi \in [0, m]$$

with $M_6 = M_6(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3)$, which lead to (4.48) owing to the arbitrariness of $\varepsilon \in (0, -\zeta_-/2)$. Furthermore, we get from (4.51), (3.65) and (3.67) that

$$\varepsilon^{-\lambda/2-1} \|\tilde{a}\|_{1, \beta; (-1/8, 1/8) \times (0, m)} + \|\tilde{h}\|_{1, \beta; (-1/8, 1/8) \times (0, m)} \leq M_7, \quad \|\tilde{e}\|_{1, \beta; (-1/8, 1/8)} \leq M_7 \varepsilon^{\lambda/2+1}$$

with $M_7 = M_7(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \delta_4)$. Using the Schauder estimates again yields

$$\|\tilde{q}\|_{2, \beta; (-1/16, 1/16) \times (0, m)} \leq M_8 (\|\tilde{q}\|_{L^\infty((-1/8, 1/8) \times (0, m))} + \|\tilde{e}\|_{1, \beta; (-1/8, 1/8)}) \leq M_9 \varepsilon^{\lambda/2+1},$$

where $M_i = M_i(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \delta_4)$ for $i = 8, 9$. Therefore,

$$\left| \frac{\partial^2 q}{\partial \varphi^2}(-\varepsilon, \psi) \right| \leq M_{10}, \quad \left| \frac{\partial^2 q}{\partial \varphi \partial \psi}(-\varepsilon, \psi) \right| \leq M_{10} \varepsilon^{\lambda/4+1/2}, \quad \left| \frac{\partial^2 q}{\partial \psi^2}(-\varepsilon, \psi) \right| \leq M_{10} \varepsilon^{\lambda/2+1}, \quad \psi \in [0, m]$$

with $M_{10} = M_{10}(\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \delta_4)$, which yield (4.49) owing to the arbitrariness of $\varepsilon \in (0, -\zeta_-/2)$. \square

5 Sonic-supersonic flows in the divergent part of the nozzle

In this section, we will establish the well-posedness of sonic-supersonic flows in the divergent part of the nozzle. For convenience, λ_+ and $\delta_{k,+}$ will be abbreviated by λ and δ_k for $k = 1, 2, 3, 4$, respectively.

We will solve the problem (3.47)–(3.52) by a fixed point argument. Let \mathcal{Q}_+ be given in advance such that $\mathcal{Q}_+ \in C([0, l_+])$ satisfies

$$c_* \leq \mathcal{Q}_+(x) \leq \bar{c} = \frac{c_* + \sqrt{2/(\gamma-1)}}{2}, \quad 0 \leq x \leq l_+. \quad (5.1)$$

Due to (5.1) and the first formula in (3.41), it holds that

$$\begin{aligned} c_* &\leq \Phi'_+(x) \leq \bar{c}\sqrt{\delta_2^2 + 1}, \quad 0 \leq x \leq l_+, \\ \frac{\delta_1}{\bar{c}^\lambda(\delta_2^2 + 1)^{\lambda/2+1}}\varphi^\lambda &\leq \Theta'_+(X_+(\varphi)) \leq \frac{\delta_2}{c_*^\lambda}\varphi^\lambda, \quad 0 \leq \varphi \leq \zeta_+. \end{aligned}$$

5.1 An iteration scheme

The existence of solutions to the problem (3.47)–(3.52) will be proved by using the Schauder fixed point theorem. For given \mathcal{Q}_+ and \tilde{Q} in some suitable set, we solve the following problem

$$Q_{\varphi\varphi} - (b(\tilde{Q})Q_\psi)_\psi = 0, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.2)$$

$$Q(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.3)$$

$$Q_\varphi(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.4)$$

$$Q_\psi(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (5.5)$$

$$Q_\psi(\varphi, m) = -\frac{\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{b(\tilde{Q}(\varphi, m))A_+^{-1}(\tilde{Q}(\varphi, m))}, \quad \varphi \in (0, \zeta_+). \quad (5.6)$$

If the problem (5.2)–(5.6) admits a unique solution Q , then we can define a mapping as

$$J((\mathcal{Q}_+, \tilde{Q})) = (A_+^{-1}(Q(\Phi_+(\cdot), m)), Q(\cdot, \cdot)).$$

A fixed point of the mapping J is a desired solution to the problem (3.47)–(3.52). Note that the problem (5.2)–(5.6) is equivalent to

$$W_\varphi + b^{1/2}(\tilde{Q})W_\psi = \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(W + Z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.7)$$

$$Z_\varphi - b^{1/2}(\tilde{Q})Z_\psi = -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}(W + Z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.8)$$

$$W(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.9)$$

$$Z(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.10)$$

$$W(\varphi, 0) + Z(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (5.11)$$

$$W(\varphi, m) + Z(\varphi, m) = \tilde{h}(\varphi), \quad \varphi \in (0, \zeta_+), \quad (5.12)$$

$$Q_\varphi(\varphi, \psi) = \frac{1}{2}(W(\varphi, \psi) - Z(\varphi, \psi)), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.13)$$

$$Q(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.14)$$

where

$$\tilde{W}(\varphi, \psi) = \tilde{Q}_\varphi(\varphi, \psi) - b^{1/2}(\tilde{Q}(\varphi, \psi))\tilde{Q}_\psi(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.15)$$

$$\tilde{Z}(\varphi, \psi) = -\tilde{Q}_\varphi(\varphi, \psi) - b^{1/2}(\tilde{Q}(\varphi, \psi))\tilde{Q}_\psi(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.16)$$

$$\tilde{h}(\varphi) = \frac{2\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{b^{1/2}(\tilde{Q}(\varphi, m))A_+^{-1}(\tilde{Q}(\varphi, m))}, \quad \varphi \in (0, \zeta_+). \quad (5.17)$$

Remark 5.1 For the system (5.7), (5.8) and (5.13), the compatibility condition is of the form

$$Q_\psi(\varphi, \psi) = -\frac{1}{2}b^{-1/2}(\tilde{Q}(\varphi, \psi))(W(\varphi, \psi) + Z(\varphi, \psi)), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.18)$$

It turns out to be more convenient to solve the problem (5.7)–(5.14) instead of the problem (5.2)–(5.6). This will be done by solving first the problem (5.7)–(5.12) and then the problem (5.13), (5.14). The problem (5.7)–(5.12) can be solved by using the contraction mapping theorem. For given (w, z) in some suitable set, we solve the following problem

$$W_\varphi + b^{1/2}(\tilde{Q})W_\psi = \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(W + z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.19)$$

$$Z_\varphi - b^{1/2}(\tilde{Q})Z_\psi = -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}(w + Z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.20)$$

$$W(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.21)$$

$$Z(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.22)$$

$$W(\varphi, 0) + Z(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (5.23)$$

$$W(\varphi, m) + Z(\varphi, m) = \tilde{h}(\varphi), \quad \varphi \in (0, \zeta_+) \quad (5.24)$$

and obtain a solution $(W, Z) = \mathcal{T}((w, z))$. Then, the unique fixed point of the contraction mapping \mathcal{T} is the unique solution to the problem (5.7)–(5.12).

In the following two subsections, we will solve the problem (5.7)–(5.14), and consequently, give a solution to the problem (3.47)–(3.52).

5.2 A hyperbolic system with singularity

Fix $\mathcal{Q}_+ \in C([0, l_+])$ satisfying (5.1). Assume that $\tilde{Q} \in C^{0,1}([0, \zeta_+] \times [0, m])$ is given and satisfies

$$-\mu_2\varphi^{\lambda+2} \leq \tilde{Q}(\varphi, \psi) \leq -\mu_1\varphi^{\lambda+2} \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.25)$$

$$-\beta_2\varphi^{\lambda+1} \leq \tilde{Q}_\varphi(\varphi, \psi) \leq -\beta_1\varphi^{\lambda+1}, \quad |\tilde{Q}_\psi(\varphi, \psi)| \leq \beta_3\varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.26)$$

where $0 < \mu_1 \leq \mu_2$, $0 < \beta_1 \leq \beta_2$, $\beta_3 > 0$. Here and what follows, μ_i , β_i , τ , κ and τ_i will denote generic positive constants. Owing to $\lambda > 2$, (3.41), (5.25) and (5.26), there exists $\tau \in (0, 1]$ such that if $0 < \zeta_+ \leq \tau$, then

$$\mu_3\varphi^{5\lambda/4+1/2} \leq \tilde{h}(\varphi) \leq \mu_4\varphi^{5\lambda/4+1/2}, \quad \tilde{h}'(\varphi) \geq 0, \quad \varphi \in (0, \zeta_+), \quad (5.27)$$

$$-\mu_5\beta_2\varphi^{\lambda+1} \leq \tilde{W}(\varphi, \psi) \leq 0, \quad 0 \leq \tilde{Z}(\varphi, \psi) \leq \mu_5\beta_2\varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.28)$$

$$-\mu_5\beta_2\varphi^{-1} \leq \frac{1}{4}b^{-1}(\tilde{Q}(\varphi, \psi))p(\tilde{Q}(\varphi, \psi))\tilde{W}(\varphi, \psi) \leq 0, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.29)$$

$$0 \leq \frac{1}{4}b^{-1}(\tilde{Q}(\varphi, \psi))p(\tilde{Q}(\varphi, \psi))\tilde{Z}(\varphi, \psi) \leq \mu_5\beta_2\varphi^{-1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.30)$$

where the constants μ_i ($i = 3, 4, 5$) ($\mu_3 \leq \mu_4$) depend on γ , m , λ , δ_1 and δ_2 , while τ also on β_1 and β_3 . The problem (5.7)–(5.14) will be solved under the assumptions (5.25)–(5.30) in this subsection.

Remark 5.2 *The constants μ_1 , μ_2 , β_1 , β_2 and β_3 , which will be determined in the proof of the existence theorem (Theorem 5.1), depend only on γ , m , λ , δ_1 and δ_2 . Therefore, the second condition in (3.41), which is only used to guarantee the second estimate in (5.27), can be relaxed as noted in Remark 3.4.*

As mentioned in the end of the last subsection, we first consider the problem (5.19)–(5.24). Note that the problem (5.19)–(5.24) can be decomposed into two problems: one is of homogeneous source terms and the other is of homogeneous boundary conditions, which are solved by the following three lemmas. For $(w, z) \in \mathbb{R}^2$, we always use $|(w, z)|$ to denote $\max\{|w|, |z|\}$ in this section.

Lemma 5.1 *The problem*

$$W_\varphi + b^{1/2}(\tilde{Q})W_\psi = 0, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.31)$$

$$Z_\varphi - b^{1/2}(\tilde{Q})Z_\psi = 0, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.32)$$

$$W(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.33)$$

$$Z(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.34)$$

$$W(\varphi, 0) + Z(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (5.35)$$

$$W(\varphi, m) + Z(\varphi, m) = \tilde{h}(\varphi), \quad \varphi \in (0, \zeta_+) \quad (5.36)$$

admits a unique weak solution $(W, Z) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m))$ satisfying

$$\lim_{\varphi \rightarrow 0^+} \| |(W, Z)(\varphi, \cdot)| \|_{L^\infty((0, m))} = 0. \quad (5.37)$$

Furthermore, (W, Z) satisfies

$$-\mu_7\varphi^{\lambda+1} \leq W(\varphi, \psi) \leq -\mu_6\varphi^{\lambda+1}, \quad \mu_6\varphi^{\lambda+1} \leq Z(\varphi, \psi) \leq \mu_7\varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m)$$

and

$$0 \leq W(\varphi, \psi) + Z(\varphi, \psi) \leq \mu_4\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where μ_6 and μ_7 ($\mu_6 \leq \mu_7$) depend only on γ , m , λ , μ_1 , μ_2 , μ_3 and μ_4 .

Proof. First consider the uniqueness. Let (W_1, Z_1) and (W_2, Z_2) be two weak solutions to the problem (5.31)–(5.36) satisfying (5.37). Set

$$(W, Z)(\varphi, \psi) = (W_1 - W_2, Z_1 - Z_2)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Then, (W, Z) satisfies

$$\lim_{\varepsilon \rightarrow 0^+} \| |(W, Z)(\varepsilon, \cdot)| \|_{L^\infty((0, m))} = 0$$

and is a weak solution to the following problem

$$U_\varphi + b^{1/2}(\tilde{Q})U_\psi = 0, \quad (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m),$$

for any $0 < \varepsilon < \zeta_+$. Hence

Letting $\varepsilon \rightarrow 0^+$ yields

i.e.

We now turn to the existence. For any $0 < \varepsilon < \zeta_+$, it follows from the classical theory for strictly hyperbolic systems that the problem

$$W^\varepsilon(\varphi, m) + Z^\varepsilon(\varphi, m) = \tilde{h}(\varphi), \quad \varphi \in (\varepsilon, \zeta_+) \quad (5.43)$$

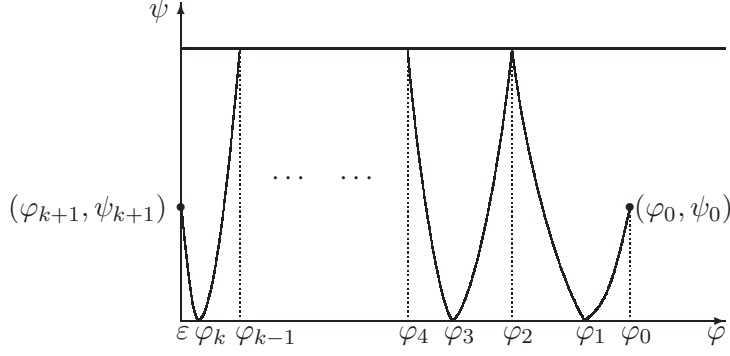
admits a unique weak solution $(W^\varepsilon, Z^\varepsilon) \in C^{0,1}([\varepsilon, \zeta_+] \times [0, m]) \times C^{0,1}([\varepsilon, \zeta_+] \times [0, m])$. Moreover, it is clear from (5.27) that

$$Z^\varepsilon(\varphi, \psi) \geq 0, \quad Z_\varphi^\varepsilon(\varphi, \psi) \geq 0, \quad Z_\psi^\varepsilon(\varphi, \psi) \geq 0, \quad (\varphi, \psi) \in [\varepsilon, \zeta_+] \times [0, m]. \quad (5.45)$$

For any $0 < \varepsilon_1 < \varepsilon_2 < \zeta_+$, it is not hard to show that

Set

the case that k is even



the case that k is odd

Fix $(\varphi_0, \psi_0) \in (\varepsilon, \zeta_+) \times (0, m)$. We estimate $W(\varphi_0, \psi_0)$ by the method of characteristics. Let $\psi = \Psi_1(\varphi)$ be the positive characteristic across (φ_0, ψ_0) , which approaches either the initial boundary $\{\varepsilon\} \times [0, m]$ or the lower boundary $(\varepsilon, \zeta_+) \times \{0\}$ at a point (φ_1, ψ_1) . If $\varphi_1 > \varepsilon$, then there exists a negative characteristic $\psi = \Psi_2(\varphi)$ from (φ_1, ψ_1) , which approaches either the initial boundary $\{\varepsilon\} \times [0, m]$ or the upper boundary $(\varepsilon, \zeta_+) \times \{m\}$ at a point (φ_2, ψ_2) . If $\varphi_2 > \varepsilon$, then there exists a positive characteristic $\psi = \Psi_3(\varphi)$ from (φ_2, ψ_2) , which approaches either the initial boundary $\{\varepsilon\} \times [0, m]$ or the lower boundary $(\varepsilon, \zeta_+) \times \{0\}$ at a point (φ_3, ψ_3) . Since the system (5.38), (5.39) is strictly hyperbolic, there exists a nonnegative integer k such that

$$\varphi_0 > \varphi_1 > \cdots > \varphi_k > \varphi_{k+1} = \varepsilon,$$

$$\psi_j = \begin{cases} 0, & 1 \leq j \leq k \text{ and } j \text{ is odd,} \\ m, & 1 \leq j \leq k \text{ and } j \text{ is even,} \end{cases} \quad 0 \leq \psi_{k+1} \leq m,$$

$$\begin{cases} \Psi'_j(\varphi) = b^{1/2}(\tilde{Q}(\varphi, \Psi_j(\varphi))), & \varphi_j < \varphi < \varphi_{j-1}, \\ \Psi_j(\varphi_j) = \psi_j, & \Psi_j(\varphi_{j-1}) = \psi_{j-1}, \end{cases} \quad 1 \leq j \leq k+1 \text{ and } j \text{ is odd,}$$

$$\begin{cases} \Psi'_j(\varphi) = -b^{1/2}(\tilde{Q}(\varphi, \Psi_j(\varphi))), & \varphi_j < \varphi < \varphi_{j-1}, \\ \Psi_j(\varphi_j) = \psi_j, & \Psi_j(\varphi_{j-1}) = \psi_{j-1}, \end{cases} \quad 1 \leq j \leq k+1 \text{ and } j \text{ is even.}$$

Then,

$$W^\varepsilon(\varphi_0, \psi_0) = - \sum_{2 \leq 2j \leq k} \tilde{h}(\varphi_{2j}).$$

For $1 \leq j \leq k+1$, it holds that

$$\int_{\varphi_j}^{\varphi_{j-1}} b^{1/2}(\tilde{Q}(s, \Psi_j(s))) ds = |\psi_{j-1} - \psi_j| \begin{cases} \in (0, m), & \text{if } j = 1, \\ = m, & \text{if } 2 \leq j \leq k, \\ \in (0, m], & \text{if } j = k+1. \end{cases}$$

Thus, it follows from this and (5.25) that

$$\varphi_{2j}^{-\lambda/4-1/2}(\varphi_{2j} - \varphi_{2(j+1)}) \leq \int_{\varphi_{2(j+1)}}^{\varphi_{2j}} s^{-\lambda/4-1/2} ds \leq \hat{\mu}_6 m, \quad 0 \leq 2j \leq k-1,$$

$$\varphi_{2j}^{-\lambda/4-1/2}(\varphi_{2(j-1)} - \varphi_{2j}) \geq \int_{\varphi_{2j}}^{\varphi_{2(j-1)}} s^{-\lambda/4-1/2} ds \geq \hat{\mu}_7 m, \quad 2 \leq 2j \leq k+1,$$

where $\hat{\mu}_6$ and $\hat{\mu}_7$ are positive constants depending only on γ , λ , μ_1 and μ_2 . Hence

$$W^\varepsilon(\varphi_0, \psi_0) \leq -\mu_3 \sum_{2 \leq 2j \leq k} \varphi_{2j}^{5\lambda/4+1/2} \leq -\frac{\mu_3}{\hat{\mu}_6 m} \sum_{2 \leq 2j \leq k-1} \varphi_{2j}^\lambda (\varphi_{2j} - \varphi_{2(j+1)}) \leq -\frac{\mu_3}{\hat{\mu}_6 m} \int_{\varphi_k}^{\varphi_2} t^\lambda dt$$

if $k \geq 2$,

$$W^\varepsilon(\varphi_0, \psi_0) \geq -\mu_4 \sum_{2 \leq 2j \leq k} \varphi_{2j}^{5\lambda/4+1/2} \geq -\frac{\mu_4}{\hat{\mu}_7 m} \sum_{2 \leq 2j \leq k} \varphi_{2j}^\lambda (\varphi_{2(j-1)} - \varphi_{2j}) \geq -\frac{\mu_4}{\hat{\mu}_7 m} \int_\varepsilon^{\varphi_0} t^\lambda dt$$

if $k \geq 0$.

These estimates lead to

$$-\mu_7 \varphi_0^{\lambda+1} \leq W(\varphi_0, \psi_0) \leq -\mu_6 \varphi_0^{\lambda+1}$$

with μ_6 and μ_7 ($\mu_6 \leq \mu_7$) depending only on γ , m , λ , μ_1 , μ_2 , μ_3 and μ_4 . Therefore,

$$-\mu_7 \varphi^{\lambda+1} \leq W(\varphi, \psi) \leq -\mu_6 \varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.46)$$

Similar, one can get that

$$\mu_6 \varphi^{\lambda+1} \leq Z(\varphi, \psi) \leq \mu_7 \varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.47)$$

Moreover, it follows from (5.44) and (5.45) that

$$0 \leq W^\varepsilon(\varphi, \psi) + Z^\varepsilon(\varphi, \psi) \leq \tilde{h}(\varphi), \quad (\varphi, \psi) \in [\varepsilon, \zeta_+] \times [0, m],$$

which implies

$$0 \leq W(\varphi, \psi) + Z(\varphi, \psi) \leq \mu_4 \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.48)$$

It is not hard to show that $(W, Z) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m))$ with (5.46)–(5.48) is a weak solution to the problem (5.31)–(5.36) satisfying (5.37). \square

We now turn to the problem with nonhomogeneous source terms and homogeneous boundary conditions, and consider first the strictly hyperbolic case and then the singular case.

Lemma 5.2 *Assume that $0 < \varepsilon < \zeta_+$, $W_0, Z_0 \in L^\infty((0, m))$, and $w, z \in L^\infty((\varepsilon, \zeta_+) \times (0, m))$ satisfy*

$$|(w, z)(\varphi, \psi)| \leq \kappa \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m).$$

Then, the problem

$$W_\varphi + b^{1/2}(\tilde{Q})W_\psi = \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(W + z), \quad (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m), \quad (5.49)$$

$$Z_\varphi - b^{1/2}(\tilde{Q})Z_\psi = -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}(w + Z), \quad (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m), \quad (5.50)$$

$$W(\varepsilon, \psi) = W_0(\psi), \quad \psi \in (0, m), \quad (5.51)$$

$$Z(\varepsilon, \psi) = Z_0(\psi), \quad \psi \in (0, m), \quad (5.52)$$

$$W(\varphi, 0) + Z(\varphi, 0) = 0, \quad \varphi \in (\varepsilon, \zeta_+), \quad (5.53)$$

$$W(\varphi, m) + Z(\varphi, m) = 0, \quad \varphi \in (\varepsilon, \zeta_+) \quad (5.54)$$

admits a unique solution $(W, Z) \in L^\infty((\varepsilon, \zeta_+) \times (0, m)) \times L^\infty((\varepsilon, \zeta_+) \times (0, m))$. Moreover, (W, Z) satisfies

$$|(W, Z)(\varphi, \psi)| \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2} \kappa \varphi^{5\lambda/4+1/2} + \| (W_0, Z_0) \|_{L^\infty((0, m))}, \quad (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m).$$

Proof. According to the classical theory for strictly hyperbolic systems, the problem (5.49)–(5.54) admits a unique weak solution $(W, Z) \in L^\infty((\varepsilon, \zeta_+) \times (0, m)) \times L^\infty((\varepsilon, \zeta_+) \times (0, m))$.

We first estimate W along a positive characteristic. Assume that

$$\Sigma_+ : \psi = \Psi_+(\varphi), \quad \hat{\varphi} \leq \varphi \leq \check{\varphi} \quad (\varepsilon \leq \hat{\varphi} < \check{\varphi} \leq \zeta_+)$$

is a positive characteristic of (5.49), i.e.

$$\begin{cases} \Psi'_+(\varphi) = b^{1/2}(\tilde{Q}(\varphi, \Psi_+(\varphi))), & \hat{\varphi} < \varphi < \check{\varphi}, \\ 0 < \Psi_+(\varphi) < m, & \hat{\varphi} < \varphi < \check{\varphi}. \end{cases}$$

On Σ_+ , W satisfies

$$\begin{aligned} & \frac{d}{d\varphi} W(\varphi, \Psi_+(\varphi)) - \frac{1}{4} b^{-1}(\tilde{Q}(\varphi, \Psi_+(\varphi))) p(\tilde{Q}(\varphi, \Psi_+(\varphi))) \tilde{W}(\varphi, \Psi_+(\varphi)) W(\varphi, \Psi_+(\varphi)) \\ &= \frac{1}{4} b^{-1}(\tilde{Q}(\varphi, \Psi_+(\varphi))) p(\tilde{Q}(\varphi, \Psi_+(\varphi))) \tilde{W}(\varphi, \Psi_+(\varphi)) z(\varphi, \Psi_+(\varphi)), \quad \hat{\varphi} < \varphi < \check{\varphi}, \end{aligned}$$

i.e.

$$\begin{aligned} & \left(W(\varphi, \Psi_+(\varphi)) \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} \right)' \\ &= -z(\varphi, \Psi_+(\varphi)) \left(\exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} \right)', \quad \hat{\varphi} < \varphi < \check{\varphi}. \end{aligned}$$

This leads to

$$\begin{aligned} & \left| \left(W(\varphi, \Psi_+(\varphi)) \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} \right)' \right| \\ & \leq \kappa \varphi^{5\lambda/4+1/2} \left(\exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} \right)', \quad \hat{\varphi} < \varphi < \check{\varphi}. \end{aligned}$$

Then, for any $\hat{\varphi} \leq \varphi \leq \check{\varphi}$, it holds that

$$\begin{aligned} & \left| W(\varphi, \Psi_+(\varphi)) \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} - W(\hat{\varphi}, \Psi_+(\hat{\varphi})) \right| \\ & \leq \kappa \int_{\hat{\varphi}}^{\varphi} t^{5\lambda/4+1/2} \left(\exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^t b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} \right)' dt \end{aligned}$$

$$\begin{aligned}
&= \kappa \varphi^{5\lambda/4+1/2} \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} - \kappa \hat{\varphi}^{5\lambda/4+1/2} \\
&\quad - \left(\frac{5}{4} \lambda + \frac{1}{2} \right) \kappa \int_{\hat{\varphi}}^{\varphi} t^{5\lambda/4-1/2} \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^t b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} dt,
\end{aligned}$$

which yields

$$\begin{aligned}
&(\kappa \varphi^{5\lambda/4+1/2} - |W(\varphi, \Psi_+(\varphi))|) \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} \\
&\geq \kappa \hat{\varphi}^{5\lambda/4+1/2} - |W(\hat{\varphi}, \Psi_+(\hat{\varphi}))| \\
&\quad + \left(\frac{5}{4} \lambda + \frac{1}{2} \right) \kappa \int_{\hat{\varphi}}^{\varphi} t^{5\lambda/4-1/2} \exp \left\{ -\frac{1}{4} \int_{\hat{\varphi}}^t b^{-1}(\tilde{Q}(s, \Psi_+(s))) p(\tilde{Q}(s, \Psi_+(s))) \tilde{W}(s, \Psi_+(s)) ds \right\} dt.
\end{aligned} \tag{5.55}$$

Similarly, we can estimate Z along a negative characteristic. Assume that

$$\Sigma_- : \psi = \Psi_-(\varphi), \quad \hat{\varphi} \leq \varphi \leq \check{\varphi} \quad (\varepsilon \leq \hat{\varphi} < \check{\varphi} \leq \zeta_+)$$

is a negative characteristic of (5.50), i.e.

$$\begin{cases} \Psi'_-(\varphi) = -b^{1/2}(\tilde{Q}(\varphi, \Psi_-(\varphi))), & \hat{\varphi} < \varphi < \check{\varphi}, \\ 0 < \Psi_-(\varphi) < m, & \hat{\varphi} < \varphi < \check{\varphi}. \end{cases}$$

On Σ_- , Z satisfies

$$\begin{aligned}
&\left(Z(\varphi, \Psi_-(\varphi)) \exp \left\{ \frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_-(s))) p(\tilde{Q}(s, \Psi_-(s))) \tilde{Z}(s, \Psi_-(s)) ds \right\} \right)' \\
&= -w(\varphi, \Psi_-(\varphi)) \left(\exp \left\{ \frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_-(s))) p(\tilde{Q}(s, \Psi_-(s))) \tilde{Z}(s, \Psi_-(s)) ds \right\} \right)', \quad \hat{\varphi} < \varphi < \check{\varphi},
\end{aligned}$$

which yields

$$\begin{aligned}
&\left| \left(Z(\varphi, \Psi_-(\varphi)) \exp \left\{ \frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_-(s))) p(\tilde{Q}(s, \Psi_-(s))) \tilde{Z}(s, \Psi_-(s)) ds \right\} \right)' \right| \\
&\leq \kappa \varphi^{5\lambda/4+1/2} \left(\exp \left\{ \frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_-(s))) p(\tilde{Q}(s, \Psi_-(s))) \tilde{Z}(s, \Psi_-(s)) ds \right\} \right)', \quad \hat{\varphi} < \varphi < \check{\varphi}.
\end{aligned}$$

Therefore, for any $\hat{\varphi} \leq \varphi \leq \check{\varphi}$, it holds that

$$\begin{aligned}
&(\kappa \varphi^{5\lambda/4+1/2} - |Z(\varphi, \Psi_-(\varphi))|) \exp \left\{ \frac{1}{4} \int_{\hat{\varphi}}^{\varphi} b^{-1}(\tilde{Q}(s, \Psi_-(s))) p(\tilde{Q}(s, \Psi_-(s))) \tilde{Z}(s, \Psi_-(s)) ds \right\} \\
&\geq \kappa \hat{\varphi}^{5\lambda/4+1/2} - |Z(\hat{\varphi}, \Psi_-(\hat{\varphi}))| \\
&\quad + \left(\frac{5}{4} \lambda + \frac{1}{2} \right) \kappa \int_{\hat{\varphi}}^{\varphi} t^{5\lambda/4-1/2} \exp \left\{ \frac{1}{4} \int_{\hat{\varphi}}^t b^{-1}(\tilde{Q}(s, \Psi_-(s))) p(\tilde{Q}(s, \Psi_-(s))) \tilde{Z}(s, \Psi_-(s)) ds \right\} dt.
\end{aligned} \tag{5.56}$$

Next we estimate W by the method of characteristics. As shown in the proof of Lemma 5.1, for any $(\varphi_0, \psi_0) \in (\varepsilon, \zeta_+) \times (0, m)$, there exists a nonnegative integer k such that

$$\varphi_0 > \varphi_1 > \cdots > \varphi_k > \varphi_{k+1} = \varepsilon,$$

$$\psi_j = \begin{cases} 0, & 1 \leq j \leq k \text{ and } j \text{ is odd,} \\ m, & 1 \leq j \leq k \text{ and } j \text{ is even,} \end{cases} \quad 0 \leq \psi_{k+1} \leq m,$$

$$\begin{cases} \Psi'_j(\varphi) = b^{1/2}(\tilde{Q}(\varphi, \Psi_j(\varphi))), & \varphi_j < \varphi < \varphi_{j-1}, \\ \Psi_j(\varphi_j) = \psi_j, & \Psi_j(\varphi_{j-1}) = \psi_{j-1}, \end{cases} \quad 1 \leq j \leq k+1 \text{ and } j \text{ is odd,}$$

$$\begin{cases} \Psi'_j(\varphi) = -b^{1/2}(\tilde{Q}(\varphi, \Psi_j(\varphi))), & \varphi_j < \varphi < \varphi_{j-1}, \\ \Psi_j(\varphi_j) = \psi_j, & \Psi_j(\varphi_{j-1}) = \psi_{j-1}, \end{cases} \quad 1 \leq j \leq k+1 \text{ and } j \text{ is even.}$$

One can see the figures in the proof of Lemma 5.1. Define a function $r(s)$ ($\varepsilon \leq s < \varphi$) as follows

$$r(s) = \begin{cases} -\frac{1}{4}b^{-1}(\tilde{Q}(s, \Psi_j(s)))p(\tilde{Q}(s, \Psi_j(s)))\tilde{W}(s, \Psi_j(s)), \\ \varphi_j \leq s < \varphi_{j-1}, 1 \leq j \leq k+1 \text{ and } j \text{ is odd,} \\ \frac{1}{4}b^{-1}(\tilde{Q}(s, \Psi_j(s)))p(\tilde{Q}(s, \Psi_j(s)))\tilde{Z}(s, \Psi_j(s)), \\ \varphi_j \leq s < \varphi_{j-1}, 1 \leq j \leq k+1 \text{ and } j \text{ is even.} \end{cases}$$

Then, for $1 \leq j \leq k+1$, it follows from (5.55) and (5.56) that

$$\begin{aligned} & (\kappa\varphi_{j-1}^{5\lambda/4+1/2} - |W(\varphi_{j-1}, \psi_{j-1})|)\exp\left\{\int_{\varphi_j}^{\varphi_{j-1}} r(s)ds\right\} \\ & \geq \kappa\varphi_j^{5\lambda/4+1/2} - |W(\varphi_j, \psi_j)| + \left(\frac{5}{4}\lambda + \frac{1}{2}\right)\kappa \int_{\varphi_j}^{\varphi_{j-1}} t^{5\lambda/4-1/2} \exp\left\{\int_{\varphi_j}^t r(s)ds\right\} dt, \quad \text{if } j \text{ is odd,} \\ & (\kappa\varphi_{j-1}^{5\lambda/4+1/2} - |Z(\varphi_{j-1}, \psi_{j-1})|)\exp\left\{\int_{\varphi_j}^{\varphi_{j-1}} r(s)ds\right\} \\ & \geq \kappa\varphi_j^{5\lambda/4+1/2} - |Z(\varphi_j, \psi_j)| + \left(\frac{5}{4}\lambda + \frac{1}{2}\right)\kappa \int_{\varphi_j}^{\varphi_{j-1}} t^{5\lambda/4-1/2} \exp\left\{\int_{\varphi_j}^t r(s)ds\right\} dt, \quad \text{if } j \text{ is even,} \end{aligned}$$

which are equivalent to

$$\begin{aligned} & \kappa\varphi_{j-1}^{5\lambda/4+1/2} - |W(\varphi_{j-1}, \psi_{j-1})| - \left(\frac{5}{4}\lambda + \frac{1}{2}\right)\kappa \int_{\varepsilon}^{\varphi_{j-1}} t^{5\lambda/4-1/2} \exp\left\{-\int_t^{\varphi_{j-1}} r(s)ds\right\} dt \\ & \geq \left(\kappa\varphi_j^{5\lambda/4+1/2} - |W(\varphi_j, \psi_j)| - \left(\frac{5}{4}\lambda + \frac{1}{2}\right)\kappa \int_{\varepsilon}^{\varphi_j} t^{5\lambda/4-1/2} \exp\left\{-\int_t^{\varphi_j} r(s)ds\right\} dt\right) \\ & \quad \cdot \exp\left\{-\int_{\varphi_j}^{\varphi_{j-1}} r(s)ds\right\}, \quad \text{if } j \text{ is odd,} \end{aligned} \tag{5.57}$$

$$\kappa\varphi_{j-1}^{5\lambda/4+1/2} - |Z(\varphi_{j-1}, \psi_{j-1})| - \left(\frac{5}{4}\lambda + \frac{1}{2}\right)\kappa \int_{\varepsilon}^{\varphi_{j-1}} t^{5\lambda/4-1/2} \exp\left\{-\int_t^{\varphi_{j-1}} r(s)ds\right\} dt$$

$$\begin{aligned}
&\geq \left(\kappa \varphi_j^{5\lambda/4+1/2} - |Z(\varphi_j, \psi_j)| - \left(\frac{5}{4}\lambda + \frac{1}{2} \right) \kappa \int_{\varepsilon}^{\varphi_j} t^{5\lambda/4-1/2} \exp \left\{ - \int_t^{\varphi_j} r(s) ds \right\} dt \right) \\
&\quad \cdot \exp \left\{ - \int_{\varphi_j}^{\varphi_{j-1}} r(s) ds \right\}, \quad \text{if } j \text{ is even.}
\end{aligned} \tag{5.58}$$

If k is odd, one can get from (5.57), (5.58), (5.51)–(5.54) that

$$\begin{aligned}
&\kappa \varphi_0^{5\lambda/4+1/2} - |W(\varphi_0, \psi_0)| - \left(\frac{5}{4}\lambda + \frac{1}{2} \right) \kappa \int_{\varepsilon}^{\varphi_0} t^{5\lambda/4-1/2} \exp \left\{ - \int_t^{\varphi_0} r(s) ds \right\} dt \\
&\geq \left(\kappa \varphi_{k+1}^{5\lambda/4+1/2} - |W(\varphi_{k+1}, \psi_{k+1})| - \left(\frac{5}{4}\lambda + \frac{1}{2} \right) \kappa \int_{\varepsilon}^{\varphi_{k+1}} t^{5\lambda/4-1/2} \exp \left\{ - \int_t^{\varphi_{k+1}} r(s) ds \right\} dt \right) \\
&\quad \cdot \exp \left\{ - \int_{\varphi_{k+1}}^{\varphi_0} r(s) ds \right\} \\
&= (\kappa \varepsilon^{5\lambda/4+1/2} - |W_0(\psi_{k+1})|) \exp \left\{ - \int_{\varepsilon}^{\varphi_0} r(s) ds \right\} \\
&\geq \kappa \varepsilon^{5\lambda/4+1/2} \exp \left\{ - \int_{\varepsilon}^{\varphi_0} r(s) ds \right\} - |W_0(\psi_{k+1})|.
\end{aligned} \tag{5.59}$$

Similarly, if k is even, one can get that

$$\begin{aligned}
&\kappa \varphi_0^{5\lambda/4+1/2} - |W(\varphi_0, \psi_0)| - \left(\frac{5}{4}\lambda + \frac{1}{2} \right) \kappa \int_{\varepsilon}^{\varphi_0} t^{5\lambda/4-1/2} \exp \left\{ - \int_t^{\varphi_0} r(s) ds \right\} dt \\
&\geq \kappa \varepsilon^{5\lambda/4+1/2} \exp \left\{ - \int_{\varepsilon}^{\varphi_0} r(s) ds \right\} - |Z_0(\psi_{k+1})|.
\end{aligned} \tag{5.60}$$

Then, it follows from (5.59) and (5.60) that

$$\begin{aligned}
|W(\varphi_0, \psi_0)| &\leq \kappa \varphi_0^{5\lambda/4+1/2} + \| |(W_0, Z_0)| \|_{L^\infty((0, m))} - \kappa \varepsilon^{5\lambda/4+1/2} \exp \left\{ - \int_{\varepsilon}^{\varphi_0} r(s) ds \right\} \\
&\quad - \left(\frac{5}{4}\lambda + \frac{1}{2} \right) \kappa \int_{\varepsilon}^{\varphi_0} t^{5\lambda/4-1/2} \exp \left\{ - \int_t^{\varphi_0} r(s) ds \right\} dt \\
&\leq \kappa \varphi_0^{5\lambda/4+1/2} + \| |(W_0, Z_0)| \|_{L^\infty((0, m))} - \kappa \varphi_0^{-\mu_5 \beta_2} \varepsilon^{5\lambda/4+1/2+\mu_5 \beta_2} \\
&\quad - \left(\frac{5}{4}\lambda + \frac{1}{2} \right) \kappa \int_{\varepsilon}^{\varphi_0} t^{5\lambda/4-1/2} \left(\frac{\varphi_0}{t} \right)^{-\mu_5 \beta_2} dt \\
&\leq \kappa \varphi_0^{5\lambda/4+1/2} + \| |(W_0, Z_0)| \|_{L^\infty((0, m))} - \kappa \varphi_0^{-\mu_5 \beta_2} \varepsilon^{5\lambda/4+1/2+\mu_5 \beta_2} \\
&\quad - \frac{5\lambda + 2}{5\lambda + 2 + 4\mu_5 \beta_2} \kappa (\varphi_0^{5\lambda/4+1/2} - \varphi_0^{-\mu_5 \beta_2} \varepsilon^{5\lambda/4+1/2+\mu_5 \beta_2}) \\
&\leq \frac{4\mu_5 \beta_2}{5\lambda + 2 + 4\mu_5 \beta_2} \kappa \varphi_0^{5\lambda/4+1/2} + \| |(W_0, Z_0)| \|_{L^\infty((0, m))}.
\end{aligned}$$

The estimate of Z can be proved similarly. □

Lemma 5.3 Assume that $(w, z) \in L^\infty((0, \zeta_+) \times (0, m))$ satisfies

$$|(w, z)(\varphi, \psi)| \leq \kappa \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \tag{5.61}$$

Then, the problem

$$W_\varphi + b^{1/2}(\tilde{Q})W_\psi = \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(W + z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.62)$$

$$Z_\varphi - b^{1/2}(\tilde{Q})Z_\psi = -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}(w + Z), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.63)$$

$$W(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.64)$$

$$Z(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.65)$$

$$W(\varphi, 0) + Z(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (5.66)$$

$$W(\varphi, m) + Z(\varphi, m) = 0, \quad \varphi \in (0, \zeta_+) \quad (5.67)$$

admits a unique solution $(W, Z) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m))$ satisfying

$$\lim_{\varphi \rightarrow 0^+} \| |(W, Z)(\varphi, \cdot)| \|_{L^\infty((0, m))} = 0. \quad (5.68)$$

Moreover, (W, Z) satisfies

$$|(W, Z)(\varphi, \psi)| \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2} \kappa \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Proof. We start with the uniqueness. Let (W_1, Z_1) and (W_2, Z_2) be two weak solutions to the problem (5.62)–(5.67) satisfying (5.68). Set

$$(W, Z)(\varphi, \psi) = (W_1 - W_2, Z_1 - Z_2)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Then, (W, Z) satisfies

$$\lim_{\varphi \rightarrow 0^+} \| |(W, Z)(\varphi, \cdot)| \|_{L^\infty((0, m))} = 0$$

and is a weak solution to the following problem

$$\begin{aligned} U_\varphi + b^{1/2}(\tilde{Q})U_\psi &= \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}U, & (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m), \\ V_\varphi - b^{1/2}(\tilde{Q})V_\psi &= -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}V, & (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m), \\ U(0, \psi) &= W(\varepsilon, \psi), & \psi \in (0, m), \\ V(0, \psi) &= Z(\varepsilon, \psi), & \psi \in (0, m), \\ U(\varphi, 0) + V(\varphi, 0) &= 0, & \varphi \in (\varepsilon, \zeta_+), \\ U(\varphi, m) + V(\varphi, m) &= 0, & \varphi \in (\varepsilon, \zeta_+) \end{aligned}$$

for any $0 < \varepsilon < \zeta_+$. It follows from Lemma 5.2 that

$$|(W, Z)(\varphi, \psi)| \leq \| |(W, Z)(\varepsilon, \cdot)| \|_{L^\infty((0, m))}, \quad 0 < \varepsilon < \varphi < \zeta_+, \quad \psi \in (0, m).$$

Letting $\varepsilon \rightarrow 0^+$ yields

$$(W, Z)(\varphi, \psi) = (0, 0), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

i.e.

$$(W_1, Z_1)(\varphi, \psi) = (W_2, Z_2)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

We now turn to the existence. For any $0 < \varepsilon < \zeta_+$, it follows from Lemma 5.2 that the following problem

$$\begin{aligned}
W_\varphi^\varepsilon + b^{1/2}(\tilde{Q})W_\psi^\varepsilon &= \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(W^\varepsilon + z), & (\varphi, \psi) &\in (\varepsilon, \zeta_+) \times (0, m), \\
Z_\varphi^\varepsilon - b^{1/2}(\tilde{Q})Z_\psi^\varepsilon &= -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}(w + Z^\varepsilon), & (\varphi, \psi) &\in (\varepsilon, \zeta_+) \times (0, m), \\
W^\varepsilon(\varepsilon, \psi) &= 0, & \psi &\in (0, m), \\
Z^\varepsilon(\varepsilon, \psi) &= 0, & \psi &\in (0, m), \\
W^\varepsilon(\varphi, 0) + Z^\varepsilon(\varphi, 0) &= 0, & \varphi &\in (\varepsilon, \zeta_+), \\
W^\varepsilon(\varphi, m) + Z^\varepsilon(\varphi, m) &= 0, & \varphi &\in (\varepsilon, \zeta_+)
\end{aligned}$$

admits a unique solution $(W^\varepsilon, Z^\varepsilon) \in L^\infty((\varepsilon, \zeta_+) \times (0, m)) \times L^\infty((\varepsilon, \zeta_+) \times (0, m))$; furthermore, $(W^\varepsilon, Z^\varepsilon)$ satisfies

$$|(W^\varepsilon, Z^\varepsilon)(\varphi, \psi)| \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2} \kappa \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (\varepsilon, \zeta_+) \times (0, m). \quad (5.69)$$

Define

$$(W^\varepsilon, Z^\varepsilon)(\varphi, \psi) = (0, 0), \quad (\varphi, \psi) \in (0, \varepsilon) \times (0, m).$$

Then, there exists a decreasing sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0, \zeta_+)$, which tends to 0 as $n \rightarrow \infty$, and $W, Z \in L^\infty((0, \zeta_+) \times (0, m))$ such that

$$(W^{\varepsilon_n}, Z^{\varepsilon_n}) \rightharpoonup (W, Z) \text{ weakly }^* \text{ in } L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m)) \text{ as } n \rightarrow \infty.$$

Moreover, it follows from (5.69) that

$$|(W, Z)(\varphi, \psi)| \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2} \kappa \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.70)$$

It is not hard to show that (W, Z) with (5.70) is a weak solution to the problem (5.62)–(5.67) satisfying (5.68). \square

Remark 5.3 In Lemma 5.3, if (5.61) is replaced by

$$|w(\varphi, \psi)| \leq \kappa \varphi^\alpha, \quad |z(\varphi, \psi)| \leq \kappa \varphi^\alpha, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m)$$

with a positive constant α . then the solution (W, Z) to the problem (5.62)–(5.67) satisfies

$$|(W, Z)(\varphi, \psi)| \leq \frac{\mu_5\beta_2}{\alpha + \mu_5\beta_2} \kappa \varphi^\alpha, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

To solve the problem (5.19)–(5.24), we introduce a norm

$$\|(w, z)\|_{\mathcal{B}} = \sup \left\{ \varphi^{-5\lambda/4-1/2} |(w, z)(\varphi, \psi)| : (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \right\}$$

and define

$$\mathcal{B} = \left\{ (w, z) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m)) : \|(w - W^*, z - Z^*)\|_{\mathcal{B}} < +\infty \right\},$$

where (W^*, Z^*) is the unique weak solution to the problem (5.31)–(5.36) satisfying (5.37). It follows from Lemma 5.1 that

$$-\mu_7\varphi^{\lambda+1} \leq W^*(\varphi, \psi) \leq -\mu_6\varphi^{\lambda+1}, \quad \mu_6\varphi^{\lambda+1} \leq Z^*(\varphi, \psi) \leq \mu_7\varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \quad (5.71)$$

and

$$0 \leq W^*(\varphi, \psi) + Z^*(\varphi, \psi) \leq \mu_4\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.72)$$

Then we have

Proposition 5.1 *For given $(w, z) \in \mathcal{B}$, the problem (5.19)–(5.24) admits a unique weak solution $(W, Z) \in \mathcal{B}$. Furthermore, if $\|(w - W^*, z - Z^*)\|_{\mathcal{B}} \leq \mu_8\beta_2$ additionally, then $\|(W - W^*, Z - Z^*)\|_{\mathcal{B}} \leq \mu_8\beta_2$, where $\mu_8 = 4\mu_4\mu_5/(5\lambda + 2)$.*

Proof. Since the uniqueness follows from a similar proof in Lemma 5.3, it suffices to consider the existence. Consider the following problem

$$\mathcal{W}_\varphi + b^{1/2}(\tilde{Q})\mathcal{W}_\psi = \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(\mathcal{W} + (W^* + Z^*) + (z - Z^*)), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.73)$$

$$\mathcal{Z}_\varphi - b^{1/2}(\tilde{Q})\mathcal{Z}_\psi = -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}((w - W^*) + (W^* + Z^*) + \mathcal{Z}), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.74)$$

$$\mathcal{W}(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.75)$$

$$\mathcal{Z}(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.76)$$

$$\mathcal{W}(\varphi, 0) + \mathcal{Z}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+), \quad (5.77)$$

$$\mathcal{W}(\varphi, m) + \mathcal{Z}(\varphi, m) = 0, \quad \varphi \in (0, \zeta_+). \quad (5.78)$$

Thanks to Lemma 5.3, the problem (5.73)–(5.78) admits a unique weak solution $(\mathcal{W}, \mathcal{Z}) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m))$ satisfying

$$|(\mathcal{W}, \mathcal{Z})(\varphi, \psi)| \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2}(\mu_4 + \|(w - W^*, z - Z^*)\|_{\mathcal{B}})\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.79)$$

Therefore, $(\mathcal{W} + W^*, \mathcal{Z} + Z^*) \in \mathcal{B}$ and it is not hard to verify that $(\mathcal{W} + W^*, \mathcal{Z} + Z^*)$ is a weak solution to the problem (5.19)–(5.24). Finally, the rest of the proposition follows from (5.79) directly. \square

We are now ready to solve the problem (5.7)–(5.14).

Proposition 5.2 *For given $\mathcal{Q}_+ \in C([0, l_+])$ with (5.1) and given $\tilde{Q} \in C^{0,1}([0, \zeta_+] \times [0, m])$ with (5.25)–(5.30), the problem (5.7)–(5.14) admits a unique weak solution (W, Z, Q) with $(W, Z) \in \mathcal{B}$. Furthermore, the following estimates hold:*

$$-\mu_7\varphi^{\lambda+1} - \mu_8\beta_2\varphi^{5\lambda/4+1/2} \leq Q_\varphi(\varphi, \psi) \leq -\mu_6\varphi^{\lambda+1} + \mu_8\beta_2\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.80)$$

$$|Q_\psi(\varphi, \psi)| \leq \mu_9(\beta_2 + 1)\varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \quad (5.81)$$

and

$$-\sigma_2\varphi^{\lambda+2} - m\mu_9(\beta_2 + 1)\varphi^{3\lambda/2+1} \leq Q(\varphi, \psi) \leq -\sigma_1\varphi^{\lambda+2} + m\mu_9(\beta_2 + 1)\varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.82)$$

where μ_9 depends only on $\gamma, \lambda, \mu_1, \mu_2, \mu_4$ and μ_8 , while σ_1 and σ_2 ($\sigma_1 \leq \sigma_2$) depend only on $\gamma, m, \lambda, \delta_1$ and δ_2 .

Proof. Let $(w, z) \in \mathcal{B}$ be given. It follows from Proposition 5.1 that the problem (5.19)–(5.24) admits a unique weak solution $(W, Z) \in \mathcal{B}$. Define the mapping

$$\mathcal{T} : \mathcal{B} \longrightarrow \mathcal{B}, \quad (w, z) \longmapsto (W, Z).$$

For given $(w_1, z_1), (w_2, z_2) \in \mathcal{B}$, it follows from the definition of \mathcal{T} that

$$\mathcal{T}((w_1, z_1))(\varphi, \psi) - \mathcal{T}((w_2, z_2))(\varphi, \psi) = (U, V)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where $(U, V) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m))$ solves the following problem

$$\begin{aligned} U_\varphi + b^{1/2}(\tilde{Q})U_\psi &= \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{W}(U + (z_1 - z_2)), & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ V_\varphi - b^{1/2}(\tilde{Q})V_\psi &= -\frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Z}((w_1 - w_2) + V), & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ U(0, \psi) &= 0, & \psi \in (0, m), \\ V(0, \psi) &= 0, & \psi \in (0, m), \\ U(\varphi, 0) + V(\varphi, 0) &= 0, & \varphi \in (0, \zeta_+), \\ U(\varphi, m) + V(\varphi, m) &= 0, & \varphi \in (0, \zeta_+) \end{aligned}$$

satisfying

$$\lim_{\varphi \rightarrow 0^+} \| (U, V)(\varphi, \cdot) \|_{L^\infty((0, m))} = 0.$$

Moreover, Lemma 5.3 leads to that (U, V) satisfies

$$|(U, V)(\varphi, \psi)| \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2} \|(w_1 - w_2, z_1 - z_2)\|_{\mathcal{B}} \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Therefore,

$$\|\mathcal{T}((w_1, z_1)) - \mathcal{T}((w_2, z_2))\|_{\mathcal{B}} \leq \frac{4\mu_5\beta_2}{5\lambda + 2 + 4\mu_5\beta_2} \|(w_1 - w_2, z_1 - z_2)\|_{\mathcal{B}},$$

which shows that \mathcal{T} is a contraction mapping on \mathcal{B} . Thus, the problem (5.7)–(5.12) admits a unique weak solution $(W, Z) \in \mathcal{B}$; furthermore, one gets from Proposition 5.1 that

$$\|(W - W^*, Z - Z^*)\|_{\mathcal{B}} \leq \mu_8\beta_2. \quad (5.83)$$

Then, we can get the weak solution $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ to the problem (5.13), (5.14). And this (W, Z, Q) is the unique weak solution to the problem (5.7)–(5.14) with $(W, Z) \in \mathcal{B}$. It follows from (5.83), (5.71) and (5.72) that

$$-\mu_7\varphi^{\lambda+1} - \mu_8\beta_2\varphi^{5\lambda/4+1/2} \leq Q_\varphi(\varphi, \psi) \leq -\mu_6\varphi^{\lambda+1} + \mu_8\beta_2\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

$$-\frac{1}{2}\mu_4\varphi^{5\lambda/4+1/2} - \mu_8\beta_2\varphi^{5\lambda/4+1/2} \leq b^{1/2}(\tilde{Q})Q_\psi(\varphi, \psi) \leq \mu_8\beta_2\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

which yield (5.80) and (5.81) immediately.

Finally, we prove (5.82). Owing to (5.7), (5.8), (5.13) and (5.18), it holds in the sense of distribution that

$$\begin{aligned} Q_{\varphi\varphi}(\varphi, \psi) &= \frac{1}{2}(W_\varphi(\varphi, \psi) - Z_\varphi(\varphi, \psi)) \\ &= -\frac{1}{2}b^{1/2}(\tilde{Q}(\varphi, \psi))(W_\psi(\varphi, \psi) + Z_\psi(\varphi, \psi)) \\ &\quad - \frac{1}{4}b^{-1/2}(\tilde{Q}(\varphi, \psi))p(\tilde{Q}(\varphi, \psi))\tilde{Q}_\psi(\varphi, \psi)(W(\varphi, \psi) + Z(\varphi, \psi)) \\ &= b^{1/2}(\tilde{Q}(\varphi, \psi))\left(b^{1/2}(\tilde{Q}(\varphi, \psi))Q_\psi(\varphi, \psi)\right)_\psi + \left(b^{1/2}(\tilde{Q}(\varphi, \psi))\right)_\psi b^{1/2}(\tilde{Q}(\varphi, \psi))Q_\psi(\varphi, \psi) \\ &= (b(\tilde{Q}(\varphi, \psi))Q_\psi(\varphi, \psi))_\psi, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \end{aligned}$$

It follows from this formula, (5.11) and (5.12) that

$$\frac{d^2}{d\varphi^2} \int_0^m Q(\varphi, \psi) d\psi = -\frac{\Theta'_+(X_+(\varphi)) \cos \Theta_+(X_+(\varphi))}{A_+^{-1}(\tilde{Q}(\varphi, m))}, \quad 0 < \varphi < \zeta_+,$$

which, together with (5.80) and (5.14), yields

$$\int_0^m Q(\varphi, \psi) d\psi = -\int_0^\varphi \int_0^t \frac{\Theta'_+(X_+(s)) \cos \Theta_+(X_+(s))}{A_+^{-1}(\tilde{Q}(s, m))} ds dt, \quad 0 < \varphi < \zeta_+.$$

Then, there exist σ_1 and σ_2 ($\sigma_1 \leq \sigma_2$) depending only on γ , m , λ , δ_1 and δ_2 such that

$$-\sigma_2\varphi^{\lambda+2} \leq \frac{1}{m} \int_0^m Q(\varphi, \psi) d\psi \leq -\sigma_1\varphi^{\lambda+2}, \quad 0 < \varphi < \zeta_+,$$

which, together with (5.81), leads to (5.82). \square

5.3 Existence of sonic-supersonic flows

Based on the well-posedness of the linearized problem (5.7)–(5.14), which is equivalent to (5.2)–(5.6), we are going to prove the existence of weak solutions to the nonlinear problem (3.47)–(3.52).

Assume that $f_+ \in C^3([0, l_+])$ satisfies (3.41) with $0 < l_+ \leq \delta_0$, where δ_0 is a positive constant to be determined below. Let $\mathcal{Q}_+ \in C([0, l_+])$ and $\mathcal{Q} \in C^{0,1}([0, 1] \times [0, m])$ be given such that (5.1) holds and

$$-2\sigma_2\left(\bar{c}\sqrt{\delta_2^2 + 1}l_+\right)^{\lambda+2}\phi^{\lambda+2} \leq \mathcal{Q}(\phi, \psi) \leq -\frac{1}{2}\sigma_1(c_*l_+)^{\lambda+2}\phi^{\lambda+2}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (5.84)$$

$$-\beta_2\left(\bar{c}\sqrt{\delta_2^2 + 1}l_+\right)^{\lambda+2}\phi^{\lambda+1} \leq \mathcal{Q}_\phi(\phi, \psi) \leq -\beta_1(c_*l_+)^{\lambda+2}\phi^{\lambda+1}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (5.85)$$

$$|\mathcal{Q}_\psi(\phi, \psi)| \leq \beta_3\left(\bar{c}\sqrt{\delta_2^2 + 1}l_+\right)^{3\lambda/2+1}\phi^{3\lambda/2+1}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (5.86)$$

where σ_1 and σ_2 are defined in Proposition 5.2, which depend only on γ , m , λ , δ_1 and δ_2 , while β_i ($i = 1, 2, 3$) ($\beta_1 \leq \beta_2$) will be determined below. Define

$$\tilde{Q}(\varphi, \psi) = \mathcal{Q}\left(\frac{\varphi}{\zeta_+}, \psi\right), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where ζ_+ is given in (3.46). Then, $\tilde{Q} \in C^{0,1}([0, \zeta_+] \times [0, m])$ satisfies

$$\begin{aligned} -2\sigma_2 \left(\frac{\bar{c}\sqrt{\delta_2^2+1}}{c_*} \right)^{\lambda+2} \varphi^{\lambda+2} &\leq \tilde{Q}(\varphi, \psi) \leq -\frac{1}{2}\sigma_1 \left(\frac{c_*}{\bar{c}\sqrt{\delta_2^2+1}} \right)^{\lambda+2} \varphi^{\lambda+2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ -\beta_2 \left(\frac{\bar{c}\sqrt{\delta_2^2+1}}{c_*} \right)^{\lambda+2} \varphi^{\lambda+1} &\leq \tilde{Q}_\varphi(\varphi, \psi) \leq -\beta_1 \left(\frac{c_*}{\bar{c}\sqrt{\delta_2^2+1}} \right)^{\lambda+2} \varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ |\tilde{Q}_\psi(\varphi, \psi)| &\leq \beta_3 \left(\frac{\bar{c}\sqrt{\delta_2^2+1}}{c_*} \right)^{3\lambda/2+1} \varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \end{aligned}$$

It follows from these estimates and (3.41) that there exists $\tau_1 \in (0, 1]$ such that if $0 < \zeta_+ \leq \tau_1$, then

$$\begin{aligned} \mu_3 \varphi^{5\lambda/4+1/2} &\leq \tilde{h}(\varphi) \leq \mu_4 \varphi^{5\lambda/4+1/2}, \quad \tilde{h}'(\varphi) \geq 0, \quad \varphi \in (0, \zeta_+), \\ -\mu_5 \beta_2 \varphi^{\lambda+1} &\leq \tilde{W}(\varphi, \psi) \leq 0, \quad 0 \leq \tilde{Z}(\varphi, \psi) \leq \mu_5 \beta_2 \varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ -\mu_5 \beta_2 \varphi^{-1} &\leq \frac{1}{4} b^{-1} (\tilde{Q}(\varphi, \psi)) p(\tilde{Q}(\varphi, \psi)) \tilde{W}(\varphi, \psi) \leq 0, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ 0 &\leq \frac{1}{4} b^{-1} (\tilde{Q}(\varphi, \psi)) p(\tilde{Q}(\varphi, \psi)) \tilde{Z}(\varphi, \psi) \leq \mu_5 \beta_2 \varphi^{-1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \end{aligned}$$

where \tilde{h} , \tilde{W} and \tilde{Z} are defined by (5.17), (5.15) and (5.16), respectively, μ_i ($i = 3, 4, 5$) ($\mu_3 \leq \mu_4$) depend on γ , m , λ , δ_1 and δ_2 , while τ_1 also on β_1 and β_3 . According to Proposition 5.2, the problem (5.7)–(5.14) admits a unique weak solution (W, Z, Q) with $(W, Z) \in \mathcal{B}$ if $0 < \zeta_+ \leq \tau_1$; furthermore, Q satisfies (5.80)–(5.82). That is to say, the problem (5.2)–(5.6) admits a unique weak solution Q with $(W, Z) \in \mathcal{B}$ if $0 < \zeta_+ \leq \tau_1$, and Q satisfies

$$-\mu_7 \varphi^{\lambda+1} - \mu_8 \beta_2 \varphi^{5\lambda/4+1/2} \leq Q_\varphi(\varphi, \psi) \leq -\mu_6 \varphi^{\lambda+1} + \mu_8 \beta_2 \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.87)$$

$$|Q_\psi(\varphi, \psi)| \leq \mu_9 (\beta_2 + 1) \varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \quad (5.88)$$

and

$$-\sigma_2 \varphi^{\lambda+2} - m \mu_9 (\beta_2 + 1) \varphi^{3\lambda/2+1} \leq Q(\varphi, \psi) \leq -\sigma_1 \varphi^{\lambda+2} + m \mu_9 (\beta_2 + 1) \varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.89)$$

where μ_i ($i = 6, 7, 8, 9$) ($\mu_6 \leq \mu_7$) depend only on γ , m , λ , δ_1 and δ_2 . Choose

$$\beta_1 = \frac{1}{2} \mu_6, \quad \beta_2 = 2\mu_7, \quad \beta_3 = \mu_9 (\beta_2 + 1) \quad (5.90)$$

and

$$\tau_2 = \min \left\{ \left(\frac{\mu_6}{2\mu_8 \beta_2} \right)^{4/(\lambda-2)}, \left(\frac{\sigma_1}{2m\mu_9 (\beta_2 + 1)} \right)^{2/(\lambda-2)}, \left(\frac{-A(\bar{c})}{2\sigma_2} \right)^{1/(\lambda+2)} \right\}, \quad (5.91)$$

which all depend only on γ , m , λ , δ_1 and δ_2 . Then, if $0 < \zeta_+ \leq \tau_2$, it follows from (5.87)–(5.91) that

$$-\beta_2 \varphi^{\lambda+1} \leq Q_\varphi(\varphi, \psi) \leq -\beta_1 \varphi^{\lambda+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.92)$$

$$|Q_\psi(\varphi, \psi)| \leq \beta_3 \varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.93)$$

$$-2\sigma_2\varphi^{\lambda+2} \leq Q(\varphi, \psi) \leq -\frac{1}{2}\sigma_1\varphi^{\lambda+2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \quad (5.94)$$

and

$$c_* \leq A_+^{-1}(Q(\varphi, \psi)) \leq \bar{c}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m). \quad (5.95)$$

Set

$$\hat{\mathcal{Q}}_+(x) = A_+^{-1}(Q(\Phi_+(x), m)), \quad 0 \leq x \leq l_+$$

and

$$\hat{\mathcal{Q}}(\phi, \psi) = Q(\zeta_+\phi, \psi), \quad (\phi, \psi) \in (0, 1) \times (0, m).$$

Due to (5.92)–(5.95), one can get that $\hat{\mathcal{Q}}_+ \in C([0, l_+])$ satisfies

$$c_* \leq \hat{\mathcal{Q}}_+(x) \leq \bar{c}, \quad 0 \leq x \leq l_+, \quad (5.96)$$

and $\hat{\mathcal{Q}} \in C^{0,1}([0, 1] \times [0, m])$ satisfies

$$-2\sigma_2\left(\bar{c}\sqrt{\delta_2^2 + 1}l_+\right)^{\lambda+2}\phi^{\lambda+2} \leq \hat{\mathcal{Q}}(\phi, \psi) \leq -\frac{1}{2}\sigma_1(c_*l_+)^{\lambda+2}\phi^{\lambda+2}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (5.97)$$

$$-\beta_2\left(\bar{c}\sqrt{\delta_2^2 + 1}l_+\right)^{\lambda+2}\phi^{\lambda+1} \leq \hat{\mathcal{Q}}_\phi(\phi, \psi) \leq -\beta_1(c_*l_+)^{\lambda+2}\phi^{\lambda+1}, \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad (5.98)$$

$$\left|\hat{\mathcal{Q}}_\psi(\phi, \psi)\right| \leq \beta_3\left(\bar{c}\sqrt{\delta_2^2 + 1}l_+\right)^{3\lambda/2+1}\phi^{3\lambda/2+1}, \quad (\phi, \psi) \in (0, 1) \times (0, m). \quad (5.99)$$

Now we choose

$$\delta_0 = \frac{1}{\bar{c}\sqrt{\delta_2^2 + 1}} \min\{\tau_1, \tau_2\}, \quad (5.100)$$

which depends only on $\gamma, m, \lambda, \delta_1$ and δ_2 . Due to (5.96)–(5.99), we get that if $0 < l_+ \leq \delta_0$, then $0 < \zeta_+ \leq \min\{\tau_1, \tau_2\}$, $\hat{\mathcal{Q}}_+ \in C([0, l_+])$ satisfies (5.1) and $\hat{\mathcal{Q}} \in C^{0,1}([0, 1] \times [0, m])$ satisfies (5.84)–(5.86) with (5.90).

For $0 < l_+ < \delta_0$, define

$$\begin{aligned} \mathcal{S} = \Big\{ (\mathcal{Q}_+, \mathcal{Q}) \in C([0, l_+]) \times C^{0,1}([0, 1] \times [0, m]) : \\ \mathcal{Q}_+ \text{ satisfies (5.1), while } \mathcal{Q} \text{ satisfies (5.84)–(5.86) with (5.90)} \Big\} \end{aligned}$$

with the norm

$$\|(\mathcal{Q}_+, \mathcal{Q})\|_{\mathcal{S}} = \max \left\{ \|\mathcal{Q}_+\|_{L^\infty((0, l_+))}, \|\mathcal{Q}\|_{L^\infty((0, 1) \times (0, m))} \right\}, \quad (\mathcal{Q}_+, \mathcal{Q}) \in \mathcal{S}.$$

Owing to the discussion above, we can define a mapping J from \mathcal{S} to itself as follows

$$J((\mathcal{Q}_+, \mathcal{Q})) = (\hat{\mathcal{Q}}_+, \hat{\mathcal{Q}}), \quad (\mathcal{Q}_+, \mathcal{Q}) \in \mathcal{S}. \quad (5.101)$$

Now the existence of sonic-supersonic flows can be stated as

Theorem 5.1 Assume that $f_+ \in C^3([0, l_+])$ satisfies (3.41). If $0 < l_+ \leq \delta_0$, then, the problem (3.47)–(3.52) admits at least one weak solution Q satisfying

$$-2\sigma_2\varphi^{\lambda+2} \leq Q(\varphi, \psi) \leq -\frac{1}{2}\sigma_1\varphi^{\lambda+2}, \quad -\beta_2\varphi^{\lambda+1} \leq Q_\varphi(\varphi, \psi) \leq -\beta_1\varphi^{\lambda+1}, \quad |Q_\psi(\varphi, \psi)| \leq \beta_3\varphi^{3\lambda/2+1},$$

$$(\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.102)$$

where δ_0 , σ_i ($i = 1, 2$) and β_i ($i = 1, 2, 3$) are defined in (5.100), Proposition 5.2 and (5.90), respectively, which all depend only on γ , m , λ , δ_1 and δ_2 .

Proof. As mentioned above, the mapping J defined by (5.101) is from \mathcal{S} to itself. It follows from (5.85), (5.86) and the embedding theorem that J is compact. Therefore, Theorem 5.1 follows from (5.92)–(5.94) and the Schauder fixed point theorem provided that J is also continuous.

It remains to show that J is continuous. Assume that $\{(\mathcal{Q}_+^{(n)}, \mathcal{Q}^{(n)})\}_{n=0}^\infty \subset \mathcal{S}$ satisfies

$$\lim_{n \rightarrow \infty} \|\mathcal{Q}_+^{(n)} - \mathcal{Q}_+^{(0)}\|_{L^\infty((0, l_+))} = 0, \quad \lim_{n \rightarrow \infty} \|\mathcal{Q}^{(n)} - \mathcal{Q}^{(0)}\|_{L^\infty((0, 1) \times (0, m))} = 0. \quad (5.103)$$

Let $\Phi_+^{(n)}$, $X_+^{(n)}$ and $\zeta_+^{(n)}$ be defined by (3.44)–(3.46) with $\mathcal{Q}_+ = \mathcal{Q}_+^{(n)}$ for $n = 0, 1, 2, \dots$ and set

$$\tilde{Q}^{(n)}(\varphi, \psi) = \mathcal{Q}^{(n)}\left(\frac{\varphi}{\zeta_+^{(n)}}, \psi\right), \quad (\varphi, \psi) \in (0, \zeta_+^{(n)}) \times (0, m), \quad n = 0, 1, 2, \dots$$

Then, (5.103) leads to

$$\lim_{n \rightarrow \infty} \|\Phi_+^{(n)} - \Phi_+^{(0)}\|_{0, 1; (0, l_+)} = 0, \quad \lim_{n \rightarrow \infty} \zeta_+^{(n)} = \zeta_+^{(0)}, \quad (5.104)$$

$$X_+^{(n)} \rightarrow X_+^{(0)} \text{ uniformly in } (0, \zeta_+^{(0)}), \quad \tilde{Q}^{(n)} \rightarrow \tilde{Q}^{(0)} \text{ uniformly in } (0, \zeta_+^{(0)}) \times (0, m). \quad (5.105)$$

It follows from the definition of J that

$$J((\mathcal{Q}_+^{(n)}, \mathcal{Q}^{(n)})) = (\hat{\mathcal{Q}}_+^{(n)}, \hat{\mathcal{Q}}^{(n)}), \quad n = 0, 1, 2, \dots,$$

where

$$\hat{\mathcal{Q}}_+^{(n)}(x) = A_+^{-1}(Q^{(n)}(\Phi_+^{(n)}(x), m)), \quad 0 \leq x \leq l_+, \quad n = 0, 1, 2, \dots$$

and

$$\hat{\mathcal{Q}}^{(n)}(\phi, \psi) = Q^{(n)}(\zeta_+^{(n)}\phi, \psi), \quad (\phi, \psi) \in (0, 1) \times (0, m), \quad n = 0, 1, 2, \dots$$

with $Q^{(n)}$ being the unique weak solution to the following problem

$$Q_{\varphi\varphi}^{(n)} - (b(\tilde{Q}^{(n)})Q_\psi^{(n)})_\psi = 0, \quad (\varphi, \psi) \in (0, \zeta_+^{(n)}) \times (0, m), \quad (5.106)$$

$$Q^{(n)}(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.107)$$

$$Q_\varphi^{(n)}(0, \psi) = 0, \quad \psi \in (0, m), \quad (5.108)$$

$$Q_\psi^{(n)}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta_+^{(n)}), \quad (5.109)$$

$$Q_\psi^{(n)}(\varphi, m) = -\frac{\Theta'_+(X_+^{(n)}(\varphi)) \cos \Theta_+(X_+^{(n)}(\varphi))}{b(\tilde{Q}^{(n)}(\varphi, m))A_+^{-1}(\tilde{Q}^{(n)}(\varphi, m))}, \quad \varphi \in (0, \zeta_+^{(n)}) \quad (5.110)$$

for $n = 0, 1, 2, \dots$. So, to show the continuity of J , it suffices to prove

$$\lim_{n \rightarrow \infty} \|\hat{\mathcal{Q}}^{(n)} - \hat{\mathcal{Q}}^{(0)}\|_{L^\infty((0,1) \times (0,m))} = 0, \quad (5.111)$$

since this limit, together with (5.104), also implies

$$\lim_{n \rightarrow \infty} \|\hat{\mathcal{Q}}_+^{(n)} - \hat{\mathcal{Q}}_+^{(0)}\|_{L^\infty((0,l_+))} = 0.$$

We will prove (5.111) by contradiction. Otherwise, there exists a positive number ε_0 and a subsequence of $\{\hat{\mathcal{Q}}^{(n)}\}_{n=1}^\infty$, denoted by itself for convenience, such that

$$\|\hat{\mathcal{Q}}^{(n)} - \hat{\mathcal{Q}}^{(0)}\|_{L^\infty((0,1) \times (0,m))} \geq \varepsilon_0, \quad n = 1, 2, \dots \quad (5.112)$$

Since $\hat{\mathcal{Q}}^{(n)}$ satisfies (5.84)–(5.86) for each $n = 1, 2, \dots$, there exist a subsequence of $\{\hat{\mathcal{Q}}^{(n)}\}_{n=1}^\infty$, denoted by itself again for convenience, and a function \mathcal{Q}^* with (5.84)–(5.86) such that

$$\hat{\mathcal{Q}}^{(n)} \longrightarrow \mathcal{Q}^* \text{ in } L^\infty((0,1) \times (0,m)) \text{ as } n \rightarrow \infty, \quad (5.113)$$

$$\hat{\mathcal{Q}}_\phi^{(n)} \longrightarrow \mathcal{Q}_\phi^* \text{ and } \hat{\mathcal{Q}}_\psi^{(n)} \longrightarrow \mathcal{Q}_\psi^* \text{ weakly } * \text{ in } L^\infty((0,1) \times (0,m)) \text{ as } n \rightarrow \infty. \quad (5.114)$$

Set

$$Q^*(\varphi, \psi) = \mathcal{Q}^*\left(\frac{\varphi}{\zeta_+^{(0)}}, \psi\right), \quad (\varphi, \psi) \in (0, \zeta_+^{(0)}) \times (0, m).$$

Letting $n \rightarrow \infty$ in (5.106)–(5.110) and using (5.104), (5.105), (5.113) and (5.114), one can get that Q^* solves the following problem

$$\begin{aligned} Q_{\varphi\varphi}^* - (b(\tilde{Q}^{(0)})Q_\psi^*)_\psi &= 0, & (\varphi, \psi) &\in (0, \zeta_+^{(0)}) \times (0, m), \\ Q^*(0, \psi) &= 0, & \psi &\in (0, m), \\ Q_\varphi^*(0, \psi) &= 0, & \psi &\in (0, m), \\ Q_\psi^*(\varphi, 0) &= 0, & \varphi &\in (0, \zeta_+^{(0)}), \\ Q_\psi^*(\varphi, m) &= -\frac{\Theta'_+(X_+^{(0)}(\varphi)) \cos \Theta_+(X_+^{(0)}(\varphi))}{b(\tilde{Q}^{(0)}(\varphi, m))A_+^{-1}(\tilde{Q}^{(0)}(\varphi, m))}, & \varphi &\in (0, \zeta_+^{(0)}). \end{aligned}$$

It follows from the uniqueness in Proposition 5.2 that

$$Q^*(\varphi, \psi) = Q^{(0)}(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+^{(0)}) \times (0, m),$$

which contradicts (5.112) and (5.113). Hence (5.111) holds. \square

5.4 Smooth sonic-supersonic flows

In this subsection, we establish the existence of $C^{1,1}$ sonic-supersonic flows under the assumption that $f_+ \in C^3([0, l_+])$ satisfies (3.41) and (3.71). The method is still a fixed point argument and the method of characteristics. To this end, one should establish a priori $C^{1,1}$ estimates for the linearized problem (5.7)–(5.14).

Lemma 5.4 Assume that (W, Z) is the unique weak solution to the problem (5.31)–(5.36) in Lemma 5.1. Then,

$$|(W_\varphi, Z_\varphi)(\varphi, \psi)| \leq \mu_{10}\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m)$$

with μ_{10} depending only on $\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \mu_1, \mu_2$ and β_2 .

Proof. Set

$$(U, V)(\varphi, \psi) = (W_\varphi, Z_\varphi)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Then, (U, V) is the unique weak solution to the following problem

$$\begin{aligned} U_\varphi + b^{1/2}(\tilde{Q})U_\psi &= \frac{1}{2}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Q}_\varphi U, & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ V_\varphi - b^{1/2}(\tilde{Q})V_\psi &= \frac{1}{2}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Q}_\varphi V, & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ U(0, \psi) &= 0, & \psi \in (0, m), \\ V(0, \psi) &= 0, & \psi \in (0, m), \\ U(\varphi, 0) + V(\varphi, 0) &= 0, & \varphi \in (0, \zeta_+), \\ U(\varphi, m) + V(\varphi, m) &= \tilde{h}'(\varphi), & \varphi \in (0, \zeta_+). \end{aligned}$$

Decompose (U, V) as

$$(U, V)(\varphi, \psi) = (U^1, V^1)(\varphi, \psi) + (U^2, V^2)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where (U^1, V^1) and (U^2, V^2) solve the following problems

$$\begin{aligned} U_\varphi^1 + b^{1/2}(\tilde{Q})U_\psi^1 &= 0, & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ V_\varphi^1 - b^{1/2}(\tilde{Q})V_\psi^1 &= 0, & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ U^1(0, \psi) &= 0, & \psi \in (0, m), \\ V^1(0, \psi) &= 0, & \psi \in (0, m), \\ U^1(\varphi, 0) + V^1(\varphi, 0) &= 0, & \varphi \in (0, \zeta_+), \\ U^1(\varphi, m) + V^1(\varphi, m) &= \tilde{h}'(\varphi), & \varphi \in (0, \zeta_+) \end{aligned}$$

and

$$\begin{aligned} U_\varphi^2 + b^{1/2}(\tilde{Q})U_\psi^2 &= \frac{1}{2}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Q}_\varphi(U^1 + U^2), & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ V_\varphi^2 - b^{1/2}(\tilde{Q})V_\psi^2 &= \frac{1}{2}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Q}_\varphi(V^1 + V^2), & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ U^2(0, \psi) &= 0, & \psi \in (0, m), \\ V^2(0, \psi) &= 0, & \psi \in (0, m), \\ U^2(\varphi, 0) + V^2(\varphi, 0) &= 0, & \varphi \in (0, \zeta_+), \\ U^2(\varphi, m) + V^2(\varphi, m) &= 0, & \varphi \in (0, \zeta_+), \end{aligned}$$

respectively. It follows from the same proof as for Lemma 5.1 that

$$|(U^1, V^1)(\varphi, \psi)| \leq \frac{1}{2}\mu_{10}\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where μ_{10} depends only on $\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \mu_1, \mu_2$ and β_2 . Then, Remark 5.3 gives

$$|(U^2, V^2)(\varphi, \psi)| \leq \frac{1}{2} \mu_{10} \varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Therefore,

$$|(U, V)(\varphi, \psi)| \leq \mu_{10} \varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

□

Now, we assume additionally that

$$|\tilde{Q}_{\varphi\varphi}(\varphi, \psi)| \leq \beta_4 \varphi^\lambda, \quad |\tilde{Q}_{\varphi\psi}(\varphi, \psi)| \leq \beta_4 \varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \quad (5.115)$$

with $\beta_4 \geq 1$ to be determined later. Then,

$$|(\tilde{W}_\varphi, \tilde{Z}_\varphi)(\varphi, \psi)| \leq \mu_{11} \beta_4 \varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where μ_{11} depends only on $\gamma, m, \lambda, \delta_1, \delta_2, \mu_1, \mu_2$ and β_2 .

Lemma 5.5 *Assume that (5.115) holds, $0 < \zeta_+ \leq \tau_3$, $\|(w - W^*, z - Z^*)\|_{\mathcal{B}} \leq \mu_8 \beta_2$ and*

$$|((w - W^*)_\varphi, (z - Z^*)_\varphi)(\varphi, \psi)| \leq N \varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m)$$

with a constant $N \geq 1$, where (W^, Z^*) is the unique weak solution to the problem (5.31)–(5.36) in Lemma 5.1. Then for any $(\varphi, \psi) \in (0, \zeta_+) \times (0, m)$,*

$$|((W - W^*)_\varphi, (Z - Z^*)_\varphi)(\varphi, \psi)| \leq \left(\frac{2\mu_8 \beta_2 + \mu_4}{2\beta_1} \mu_{11} \beta_4 \tau_3^{\lambda/4-1/2} + \frac{\mu_5 \beta_2}{\mu_5 \beta_2 + 2\beta_1} N + \mu_{12} \right) \varphi^\lambda, \quad (5.116)$$

where (W, Z) is the unique weak solution to the problem (5.19)–(5.24) in Proposition 5.1 and μ_{12} depends only on $\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \mu_1, \mu_2$ and β_2 .

Proof. Let $(\mathcal{W}, \mathcal{Z})$ be the weak solution to the problem (5.73)–(5.78) in the proof of Proposition 5.1. Set

$$(U, V)(\varphi, \psi) = (\mathcal{W}_\varphi, \mathcal{Z}_\varphi)(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Then, (U, V) solves the following problem

$$\begin{aligned} U_\varphi + b^{1/2}(\tilde{Q})U_\psi &= \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})(2\tilde{Q}_\varphi + \tilde{W})(U + F^1 + F^2 + F^3), & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ V_\varphi - b^{1/2}(\tilde{Q})V_\psi &= \frac{1}{4}b^{-1}(\tilde{Q})p(\tilde{Q})(2\tilde{Q}_\varphi - \tilde{Z})(V + G^1 + G^2 + G^3), & (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\ U(0, \psi) &= 0, & \psi \in (0, m), \\ V(0, \psi) &= 0, & \psi \in (0, m), \\ U(\varphi, 0) + V(\varphi, 0) &= 0, & \varphi \in (0, \zeta_+), \\ U(\varphi, m) + V(\varphi, m) &= 0, & \varphi \in (0, \zeta_+), \end{aligned}$$

where

$$F^1 = (2\tilde{Q}_\varphi + \tilde{W})^{-1} \tilde{W}_\varphi (\mathcal{W} + (W^* + Z^*) + (z - Z^*)),$$

$$\begin{aligned}
F^2 &= (2\tilde{Q}_\varphi + \tilde{W})^{-1}\tilde{W}(z - Z^*)_\varphi, \\
F^3 &= (2\tilde{Q}_\varphi + \tilde{W})^{-1}\tilde{W}\left(b(\tilde{Q})p^{-1}(\tilde{Q})(b^{-1}(\tilde{Q})p(\tilde{Q}))_\varphi - \frac{1}{2}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Q}_\varphi\right)(\mathcal{W} + (W^* + Z^*) + (z - Z^*)) \\
&\quad + (2\tilde{Q}_\varphi + \tilde{W})^{-1}\tilde{W}(W^* + Z^*)_\varphi, \\
G^1 &= -(2\tilde{Q}_\varphi - \tilde{Z})^{-1}\tilde{Z}_\varphi((w - W^*) + (W^* + Z^*) + \mathcal{Z}), \\
G^2 &= -(2\tilde{Q}_\varphi - \tilde{Z})^{-1}\tilde{Z}(w - W^*)_\varphi, \\
G^3 &= (2\tilde{Q}_\varphi - \tilde{Z})^{-1}\tilde{Z}\left(\frac{1}{2}b^{-1}(\tilde{Q})p(\tilde{Q})\tilde{Q}_\varphi - b(\tilde{Q})p^{-1}(\tilde{Q})(b^{-1}(\tilde{Q})p(\tilde{Q}))_\varphi\right)((w - W^*) + (W^* + Z^*) + \mathcal{Z}) \\
&\quad - (2\tilde{Q}_\varphi - \tilde{Z})^{-1}\tilde{Z}(W^* + Z^*)_\varphi.
\end{aligned}$$

Direct calculations show that

$$\begin{aligned}
|(F^1, G^1)(\varphi, \psi)| &\leq \frac{2\mu_8\beta_2 + \mu_4}{2\beta_1}\mu_{11}\beta_4\tau_3^{\lambda/4-1/2}\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\
|(F^2, G^2)(\varphi, \psi)| &\leq \frac{\mu_5\beta_2}{\mu_5\beta_2 + 2\beta_1}N\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m)
\end{aligned}$$

and

$$|(F^3, G^3)(\varphi, \psi)| \leq \mu_{12}\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

where μ_{12} depends only on $\gamma, m, \lambda, \delta_1, \delta_2, \delta_3, \mu_1, \mu_2$ and β_2 . Then, Remark 5.3 yields

$$|(U, V)(\varphi, \psi)| \leq \left(\frac{2\mu_8\beta_2 + \mu_4}{2\beta_1}\mu_{11}\beta_4\tau_3^{\lambda/4-1/2} + \frac{\mu_5\beta_2}{\mu_5\beta_2 + 2\beta_1}N + \mu_{12}\right)\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

□

Remark 5.4 Under the assumptions of Lemma 5.5, if, in addition,

$$\frac{2\mu_8\beta_2 + \mu_4}{2\beta_1}\mu_{11}\beta_4\tau_3^{\lambda/4-1/2} + \mu_{12} \leq \frac{2\beta_1}{\mu_5\beta_2 + 2\beta_1}N, \quad (5.117)$$

then (5.116) implies

$$|((W - W^*)_\varphi, (Z - Z^*)_\varphi)(\varphi, \psi)| \leq N\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m).$$

Therefore, replacing \mathcal{B} with

$$\begin{aligned}
\tilde{\mathcal{B}} &= \left\{ (w, z) \in L^\infty((0, \zeta_+) \times (0, m)) \times L^\infty((0, \zeta_+) \times (0, m)) : \|(w - W^*, z - Z^*)\|_{\mathcal{B}} \leq \mu_8\beta_2, \right. \\
&\quad \left. |((w - W^*)_\varphi, (z - Z^*)_\varphi)(\varphi, \psi)| \leq N\varphi^\lambda, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m) \right\}
\end{aligned}$$

in the proof of Proposition 5.2, we can show that

Proposition 5.3 Let the assumptions of Proposition 5.2 hold and (W, Z, Q) be the unique weak solution to the problem (5.7)–(5.14). Assume further that $0 < \zeta_+ \leq \tau_3$, $N \geq 1$ and (5.117) holds. Then $(W, Z) \in \tilde{\mathcal{B}}$ and

$$|Q_{\varphi\varphi}(\varphi, \psi)| \leq (N + \mu_{10})\varphi^\lambda, \quad |Q_{\varphi\psi}(\varphi, \psi)| \leq \mu_{13}(N + \mu_{10})\varphi^{5\lambda/4+1/2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.118)$$

where μ_{13} depends only on $\gamma, m, \lambda, \delta_1, \delta_2, \mu_1, \mu_2$ and β_2 .

Remark 5.5 Under the assumptions of Proposition 5.3, if, in addition,

$$\beta_4 \geq \max \left\{ (N + \mu_{10}) \left(\frac{\bar{c}\sqrt{\delta_2^2 + 1}}{c_*} \right)^{\lambda+2}, \mu_{13}(N + \mu_{10}) \left(\frac{\bar{c}\sqrt{\delta_2^2 + 1}}{c_*} \right)^{5\lambda/4+3/2} \right\}, \quad (5.119)$$

then (5.118) leads to

$$\begin{aligned} |Q_{\varphi\varphi}(\varphi, \psi)| &\leq \beta_4 \left(\frac{c_*}{\bar{c}\sqrt{\delta_2^2 + 1}} \right)^{\lambda+2} \varphi^\lambda, & (\varphi, \psi) &\in (0, \zeta_+) \times (0, m), \\ |Q_{\varphi\psi}(\varphi, \psi)| &\leq \beta_4 \left(\frac{c_*}{\bar{c}\sqrt{\delta_2^2 + 1}} \right)^{5\lambda/4+3/2} \varphi^{5\lambda/4+1/2}, & (\varphi, \psi) &\in (0, \zeta_+) \times (0, m). \end{aligned}$$

We now choose

$$\begin{aligned} \tau_3 &= \left(\frac{\beta_1^2}{\mu_{11}(1 + \mu_{13})(2\mu_8\beta_2 + \mu_4)(\mu_5\beta_2 + 2\beta_1)} \right)^{4/(\lambda-2)} \left(\frac{c_*}{\bar{c}\sqrt{\delta_2^2 + 1}} \right)^{(5\lambda+6)/(\lambda-2)}, \\ \beta_4 &= (1 + \mu_{13}) \left(2\mu_{10} + \frac{\mu_{12}(\mu_5\beta_2 + 2\beta_1)}{\beta_1} + 1 \right) \left(\frac{\bar{c}\sqrt{\delta_2^2 + 1}}{c_*} \right)^{5\lambda/4+3/2} \end{aligned}$$

and

$$N = \mu_{10} + \frac{\mu_{12}(\mu_5\beta_2 + 2\beta_1)}{\beta_1} + 1.$$

Then, (5.117) and (5.119) hold. Set

$$\tilde{\delta}_0 = \frac{1}{\bar{c}\sqrt{\delta_2^2 + 1}} \min\{\tau_1, \tau_2, \tau_3\}$$

and

$$\begin{aligned} \tilde{\mathcal{S}} &= \left\{ (\mathcal{Q}_+, \mathcal{Q}) \in C([0, l_+]) \times C^{1,1}([0, 1] \times [0, m]) : \mathcal{Q}_+ \text{ satisfies (5.1), while } \mathcal{Q} \text{ satisfies} \right. \\ &\quad (5.84)-(5.86) \text{ with (5.90), } |\mathcal{Q}_{\phi\phi}(\phi, \psi)| \leq \beta_4(c_*l_+)^{\lambda+2}\phi^\lambda \text{ and} \\ &\quad \left. |\mathcal{Q}_{\phi\psi}(\phi, \psi)| \leq \beta_4(c_*l_+)^{5\lambda/4+3/2}\phi^{5\lambda/4+1/2} \text{ for } (\phi, \psi) \in (0, 1) \times (0, m) \right\}. \end{aligned}$$

Replacing δ_0 and \mathcal{S} with $\tilde{\delta}_0$ and $\tilde{\mathcal{S}}$ in the proof of Theorem 5.1, one can prove that

Theorem 5.2 Assume that $f_+ \in C^3([0, l_+])$ satisfies (3.41) and (3.71). If $0 < l_+ \leq \tilde{\delta}_0$, then the problem (3.47)–(3.52) admits at least a solution $Q \in C^{1,1}([0, \zeta_+] \times [0, m])$, which satisfies (5.102) and

$$\begin{aligned} |Q_{\varphi\varphi}(\varphi, \psi)| &\leq \mu_{14}\varphi^\lambda, \quad |Q_{\varphi\psi}(\varphi, \psi)| \leq \mu_{14}\varphi^{5\lambda/4+1/2}, \quad |Q_{\psi\psi}(\varphi, \psi)| \leq \mu_{14}\varphi^{3\lambda/2+1}, \\ &(\varphi, \psi) \in (0, \zeta_+) \times (0, m), \end{aligned} \quad (5.120)$$

where $\tilde{\delta}_0$ and μ_{14} depend only on γ , m , λ , δ_1 , δ_2 and δ_3 .

5.5 Uniqueness of sonic-supersonic flows

In this subsection, we study the uniqueness of sonic-supersonic flows. Owing to the strong singularity of the equation at the sonic curve, one can only prove the uniqueness of sonic-supersonic flows with precise estimates as in the existence (Theorems 5.1 and 5.2). In Theorem 5.2, we get a $C^{1,1}$ sonic-supersonic flow under the assumption that $f_+ \in C^3([0, l_+])$ satisfies (3.41) and (3.71). This smooth sonic-supersonic flow will be shown to be unique if $f_+ \in C^4([0, l_+])$ additionally. Furthermore, it is also unique in the space of weak solutions given in Theorem 5.1. It is noted that the additional condition $f_+ \in C^4([0, l_+])$ arises from the nonlocal and implicit boundary condition (3.51).

Theorem 5.3 *Assume that $f_+ \in C^4([0, l_+])$ satisfies (3.41) and (3.71). Then the problem (3.47)–(3.52) admits at most one weak solution $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ satisfying*

$$-M_2\varphi^{\lambda+2} \leq Q(\varphi, \psi) \leq -M_1\varphi^{\lambda+2}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.121)$$

$$-M_2\varphi^{\lambda+1} \leq Q_\varphi(\varphi, \psi) \leq -M_1\varphi^{\lambda+1}, \quad |Q_\psi(\varphi, \psi)| \leq M_2\varphi^{3\lambda/2+1}, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \quad (5.122)$$

where M_1 and M_2 ($M_1 \leq M_2$) are positive constants.

Proof. Note that the problem (3.47)–(3.52) is equivalent to the problem (3.16)–(3.21), which has standard initial and boundary value conditions, and (3.16) is strictly hyperbolic away from the sonic curve. So, it suffices to prove the uniqueness of weak solutions to the problem (3.47)–(3.52) for small $\zeta_+ > 0$. Thanks to Theorem 5.2, for sufficiently small $\zeta_+ \in (0, 1)$, the problem (3.47)–(3.52) admits a solution $\tilde{Q} \in C^{1,1}([0, \zeta_+] \times [0, m])$ satisfying (5.102) and (5.120). Therefore, one need only to prove that \tilde{Q} is the unique weak solution to the problem (3.47)–(3.52). Assume that $\hat{Q} \in C^{0,1}([0, \zeta_+] \times [0, m])$ with (5.121) and (5.122) is a weak solution to the problem (3.47)–(3.52). Note that

$$\operatorname{div}(\hat{Q}_\varphi, -b(\hat{Q})\hat{Q}_\psi) = \operatorname{div}(\hat{Q}_\psi, -\hat{Q}_\varphi) = 0$$

in $(0, \zeta_+) \times (0, m)$ in the sense of distribution. It follows from the theory of L^∞ divergence-measure vector fields ([1]), (5.121) and (5.122) that \hat{Q}_φ and \hat{Q}_ψ have L^∞ trace on $[0, \zeta_+] \times \{\psi\}$ and $\{\varphi\} \times [0, m]$ for each $\psi \in [0, m]$ and $\varphi \in [0, \zeta_+]$.

Denote the functions given by (3.45) corresponding to \tilde{Q} and \hat{Q} by \tilde{X}_+ and \hat{X}_+ , respectively. For convenience, we use $\mu > 0$ and $M > 0$ in the proof to denote generic positive constants depending only on $\gamma, m, \lambda, l_+, \delta_1, \delta_2, \delta_3, \|f_+^{(4)}\|_{L^\infty((0, l_+))}, M_1$ and M_2 . Set

$$Q(\varphi, \psi) = \tilde{Q}(\varphi, \psi) - \hat{Q}(\varphi, \psi), \quad (\varphi, \psi) \in [0, \zeta_+] \times [0, m]$$

and

$$X(\varphi) = \tilde{X}_+(\varphi) - \hat{X}_+(\varphi), \quad (\varphi, \psi) \in [0, \zeta_+].$$

Then $Q \in C^{0,1}([0, \zeta_+] \times [0, m])$ and $X \in C^1([0, \zeta_+])$ satisfy

$$b(\tilde{Q}(\varphi, \psi)) - b(\hat{Q}(\varphi, \psi)) = h(\varphi, \psi)Q(\varphi, \psi), \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m),$$

$$Q_\psi(\varphi, m) = e_1(\varphi)X(\varphi) + e_2(\varphi)Q(\varphi, m), \quad \varphi \in (0, \zeta_+) \quad (5.123)$$

and

$$X'(\varphi) = d_1(\varphi)X(\varphi) + d_2(\varphi)Q(\varphi, m), \quad \varphi \in (0, \zeta_+), \quad (5.124)$$

where

$$\begin{aligned}
h(\varphi, \psi) &= - \int_0^1 K''(t\tilde{Q}(\varphi, \psi) + (1-t)\hat{Q}(\varphi, \psi))dt, \quad (\varphi, \psi) \in (0, \zeta_+) \times (0, m), \\
e_1(\varphi) &= - \frac{1}{b(\tilde{Q}(\varphi, m))A_+^{-1}(\tilde{Q}(\varphi, m))} \int_0^1 E_1(t\tilde{X}_+(\varphi) + (1-t)\hat{X}_+(\varphi))dt, \quad \varphi \in (0, \zeta_+), \\
e_2(\varphi) &= -\Theta'_+(\hat{X}_+(\varphi)) \cos \Theta_+(\hat{X}_+(\varphi)) \int_0^1 E_2(t\tilde{Q}(\varphi, m) + (1-t)\hat{Q}(\varphi, m))dt, \quad \varphi \in (0, \zeta_+), \\
d_1(\varphi) &= \frac{1}{A_+^{-1}(\tilde{Q}(\varphi, m))} \int_0^1 D_1(t\tilde{X}_+(\varphi) + (1-t)\hat{X}_+(\varphi))dt, \quad \varphi \in (0, \zeta_+), \\
d_2(\varphi) &= \cos \Theta_+(\hat{X}_+(\varphi)) \int_0^1 D_2(t\tilde{Q}(\varphi, m) + (1-t)\hat{Q}(\varphi, m))dt, \quad \varphi \in (0, \zeta_+)
\end{aligned}$$

with

$$E_1(s) = (\Theta'_+(s) \cos \Theta_+(s))', \quad D_1(s) = (\cos \Theta_+(s))', \quad 0 \leq s \leq l_+$$

and

$$E_2(s) = \left(\frac{1}{b(s)A_+^{-1}(s)} \right)', \quad D_2(s) = \left(\frac{1}{A_+^{-1}(s)} \right)', \quad s < 0.$$

Making use of the conditions (3.41), (3.71) and $f_+ \in C^4([0, l_+])$ on f_+ and the asymptotic behavior (5.102) and (5.120) on \tilde{Q} and (5.121) and (5.122) on \hat{Q} near $\varphi = 0$, one can show by direct calculations that for $(\varphi, \psi) \in (0, \zeta_+) \times (0, m)$,

$$\begin{aligned}
|Q_\varphi(\varphi, \psi)| &\leq M\varphi^{\lambda+1}, \quad |Q_\psi(\varphi, \psi)| \leq M\varphi^{3\lambda/2+1}, \\
\mu\varphi^{-3\lambda/2-3} &\leq h(\varphi, \psi) \leq M\varphi^{-3\lambda/2-3}, \quad -M\varphi^{-3\lambda/2-4} \leq h_\varphi(\varphi, \psi) \leq -\mu\varphi^{-3\lambda/2-4},
\end{aligned}$$

and for $\varphi \in (0, \zeta_+)$,

$$\begin{aligned}
|e_1(\varphi)| &\leq M\varphi^{3\lambda/2}, \quad |e_2(\varphi)| \leq M\varphi^{\lambda/2-1}, \quad |e'_1(\varphi)| \leq M\varphi^{\lambda/2+1}, \\
|e'_2(\varphi)| &\leq M\varphi^{\lambda/2-2}, \quad |d_1(\varphi)| \leq M\varphi^{2\lambda+1}, \quad |d_2(\varphi)| \leq M\varphi^{-\lambda/2-1}, \\
-M\varphi^{3\lambda/2+1} &\leq \tilde{Q}_\psi(\varphi, m) \leq -\mu\varphi^{3\lambda/2+1}, \quad |\tilde{Q}_{\varphi\psi}(\varphi, m)| \leq M\varphi^{3\lambda/2}.
\end{aligned}$$

Here, the bound of e'_1 depends on $\|f_+^{(4)}\|_{L^\infty((0, l_+))}$ and it is noted that this is the only reason for the assumption that $f_+ \in C^4([0, l_+])$.

First we estimate X . It follows from (5.124) and the regularity estimates of d_1 and d_2 that

$$|X(\phi)| \leq M \int_0^\phi \varphi^{-\lambda/2-1} |Q(\varphi, m)| d\varphi, \quad \phi \in (0, \zeta_+), \quad (5.125)$$

which implies

$$X^2(\phi) \leq M\phi^2 \int_0^\phi \varphi^{-\lambda-3} Q^2(\varphi, m) d\varphi, \quad \phi \in (0, \zeta_+) \quad (5.126)$$

and

$$|X'(\phi)|^2 \leq M\phi^{4\lambda+4} \int_0^\phi \varphi^{-\lambda-3} Q^2(\varphi, m) d\varphi + M\phi^{-\lambda-2} Q^2(\phi, m), \quad \phi \in (0, \zeta_+). \quad (5.127)$$

The Hölder inequality gives

$$\begin{aligned}
\sup_{(0,m)} Q^2(\phi, \cdot) &\leq \left(\frac{1}{m} \int_0^m |Q(\phi, \psi)| d\psi + \int_0^m |Q_\psi(\phi, \psi)| d\psi \right)^2 \\
&\leq \left(\frac{1}{m} \int_0^\phi \int_0^m |Q_\varphi(\varphi, \psi)| d\varphi d\psi + \int_0^m |Q_\psi(\phi, \psi)| d\psi \right)^2 \\
&\leq M\phi^{\lambda+3} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2(\varphi, \psi) d\varphi d\psi + M \int_0^m Q_\psi^2(\phi, \psi) d\psi, \quad \phi \in (0, \zeta_+). \quad (5.128)
\end{aligned}$$

Then, one gets from (5.126)–(5.128) that

$$X^2(\phi) \leq M\phi^3 \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2(\varphi, \psi) d\varphi d\psi + M\phi^{\lambda/2+2} \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2(\varphi, \psi) d\psi, \quad \phi \in (0, \zeta_+) \quad (5.129)$$

and

$$\begin{aligned}
|X'(\phi)|^2 &\leq M\phi \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2(\varphi, \psi) d\varphi d\psi + M\phi^{9\lambda/2+4} \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2(\varphi, \psi) d\psi \\
&\quad + M\phi^{-\lambda-2} \int_0^m Q_\psi^2(\phi, \psi) d\psi, \quad \phi \in (0, \zeta_+). \quad (5.130)
\end{aligned}$$

Next, we improve the estimate (5.128) for $\|Q(\phi, \cdot)\|_{L^\infty((0,m))}$. It follows from the definition of weak solutions that

$$\begin{aligned}
\frac{d}{d\psi} \int_0^m Q(\phi, \psi) d\psi &= \int_0^\phi \left(b(\tilde{Q}(\varphi, m)) \frac{\partial \tilde{Q}}{\partial \psi}(\varphi, m) - b(\hat{Q}(\varphi, m)) \frac{\partial \hat{Q}}{\partial \psi}(\varphi, m) \right) d\varphi \\
&= \int_0^\phi (e_3(\varphi) X(\varphi) + e_4(\varphi) Q(\varphi, m)) d\varphi, \quad \phi \in (0, \zeta_+),
\end{aligned}$$

where

$$\begin{aligned}
e_3(\varphi) &= -\frac{1}{A_+^{-1}(\tilde{Q}(\varphi, m))} \int_0^1 E_1(t\tilde{X}_+(\varphi) + (1-t)\hat{X}_+(\varphi)) dt, \quad \varphi \in (0, \zeta_+), \\
e_4(\varphi) &= -\Theta'_+(\hat{X}_+(\varphi)) \cos \Theta_+(\hat{X}_+(\varphi)) \int_0^1 D_2(t\tilde{Q}(\varphi, m) + (1-t)\hat{Q}(\varphi, m)) dt, \quad \varphi \in (0, \zeta_+)
\end{aligned}$$

with

$$|e_3(\varphi)| \leq M\varphi^{\lambda-1}, \quad |e_4(\varphi)| \leq M\varphi^{\lambda/2-1}, \quad \varphi \in (0, \zeta_+).$$

Therefore,

$$\begin{aligned}
\left| \int_0^m Q(\phi, \psi) d\psi \right| &\leq \phi \int_0^\phi |e_3(\varphi) X(\varphi) + e_4(\varphi) Q(\varphi, m)| d\varphi \\
&\leq M\phi^{\lambda+1/2} \left(\int_0^\phi X^2(\varphi) d\varphi \right)^{1/2} + M\phi^{\lambda/2+1/2} \left(\int_0^\phi Q^2(\varphi, m) d\varphi \right)^{1/2}, \quad \phi \in (0, \zeta_+),
\end{aligned}$$

which, together with (5.129) and (5.128), implies

$$\left(\int_0^m Q(\phi, \psi) d\psi \right)^2 \leq M\phi^{2\lambda+5} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2(\varphi, \psi) d\varphi d\psi$$

$$+ M\phi^{5\lambda/2+4} \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2(\varphi, \psi) d\psi, \quad \phi \in (0, \zeta_+).$$

Thus

$$\begin{aligned} \sup_{(0,m)} Q^2(\phi, \cdot) &\leq \left(\frac{1}{m} \left| \int_0^m Q(\phi, \psi) d\psi \right| + \int_0^m |Q_\psi(\phi, \psi)| d\psi \right)^2 \\ &\leq M\phi^{2\lambda+5} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2(\varphi, \psi) d\varphi d\psi + M\phi^{5\lambda/2+4} \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2(\varphi, \psi) d\psi \\ &\quad + M \int_0^m Q_\psi^2(\phi, \psi) d\psi, \quad \phi \in (0, \zeta_+). \end{aligned} \quad (5.131)$$

Moreover, it follows from (5.125) and (5.131) that

$$\begin{aligned} X^2(\phi) &\leq M\phi^{\lambda/2+2} \int_0^\phi \varphi^{-3\lambda/2-3} Q^2(\varphi, m) d\varphi \\ &\leq M\phi^{\lambda+5} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2(\varphi, \psi) d\varphi d\psi + M\phi^{\lambda/2+2} \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2(\varphi, \psi) d\psi, \quad \phi \in (0, \zeta_+), \end{aligned} \quad (5.132)$$

which improves the estimate (5.129).

Now we are ready to prove the theorem by a weighted energy estimate. Let $\phi \in (0, \zeta_+]$ be given. Multiplying the equation of \tilde{Q} by $\varphi^{-\lambda-1}\tilde{Q}_\varphi$ and $-\varphi^{-\lambda-1}\hat{Q}_\varphi$, respectively, and then integrating over $(0, \phi) \times (0, m)$, one can get that

$$\begin{aligned} 0 &= \int_0^\phi \int_0^m \varphi^{-\lambda-1} \tilde{Q}_{\varphi\varphi} \tilde{Q}_\varphi d\varphi d\psi - \int_0^\phi \int_0^m \varphi^{-\lambda-1} (b(\tilde{Q})\tilde{Q}_\psi)_\psi \tilde{Q}_\varphi d\varphi d\psi \\ &= \frac{1}{2} \int_0^m \varphi^{-\lambda-1} \tilde{Q}_\varphi^2 d\psi \Big|_{\varphi=\phi} + \frac{\lambda+1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-2} \tilde{Q}_\varphi^2 d\varphi d\psi - \int_0^\phi \varphi^{-\lambda-1} b(\tilde{Q}) \tilde{Q}_\varphi \tilde{Q}_\psi d\varphi \Big|_{\psi=m} \\ &\quad + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} b(\tilde{Q}) \tilde{Q}_\psi^2 d\psi \Big|_{\varphi=\phi} - \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} b(\tilde{Q}))_\varphi \tilde{Q}_\psi^2 d\varphi d\psi \end{aligned} \quad (5.133)$$

and

$$0 = - \int_0^\phi \int_0^m \varphi^{-\lambda-1} \tilde{Q}_{\varphi\varphi} \hat{Q}_\varphi d\varphi d\psi + \int_0^\phi \int_0^m \varphi^{-\lambda-1} (b(\tilde{Q})\tilde{Q}_\psi)_\psi \hat{Q}_\varphi d\varphi d\psi. \quad (5.134)$$

It follows from the definition of weak solutions and a standard limit process that

$$\begin{aligned} 0 &= \frac{1}{2} \int_0^m \varphi^{-\lambda-1} \hat{Q}_\varphi^2 d\psi \Big|_{\varphi=\phi} + \frac{\lambda+1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-2} \hat{Q}_\varphi^2 d\varphi d\psi - \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) \hat{Q}_\varphi \hat{Q}_\psi d\varphi \Big|_{\psi=m} \\ &\quad + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} b(\hat{Q}) \hat{Q}_\psi^2 d\psi \Big|_{\varphi=\phi} - \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} b(\hat{Q}))_\varphi \hat{Q}_\psi^2 d\varphi d\psi. \end{aligned} \quad (5.135)$$

and

$$0 = - \int_0^m \varphi^{-\lambda-1} \tilde{Q}_\varphi \hat{Q}_\varphi d\psi \Big|_{\varphi=\phi} - (\lambda+1) \int_0^\phi \int_0^m \varphi^{-\lambda-2} \tilde{Q}_\varphi \hat{Q}_\varphi d\varphi d\psi + \int_0^\phi \int_0^m \varphi^{-\lambda-1} \tilde{Q}_{\varphi\varphi} \hat{Q}_\varphi d\varphi d\psi$$

$$+ \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) \tilde{Q}_\varphi \hat{Q}_\psi d\varphi \Big|_{\psi=m} - \int_0^\phi \int_0^m \varphi^{-\lambda-1} b(\hat{Q}) \tilde{Q}_{\varphi\psi} \hat{Q}_\psi d\varphi d\psi. \quad (5.136)$$

A direct calculation and a standard limit process show that

$$\begin{aligned} & \int_0^\phi \int_0^m \varphi^{-\lambda-1} (b(\tilde{Q}) \tilde{Q}_\psi)_\psi \hat{Q}_\varphi d\varphi d\psi \\ &= \int_0^\phi \varphi^{-\lambda-1} b(\tilde{Q}) \tilde{Q}_\psi \hat{Q}_\varphi d\varphi \Big|_{\psi=m} - \int_0^m \varphi^{-\lambda-1} b(\tilde{Q}) \tilde{Q}_\psi \hat{Q}_\psi d\psi \Big|_{\varphi=\phi} \\ & \quad + \int_0^\phi \int_0^m (\varphi^{-\lambda-1} b(\tilde{Q}))_\varphi \tilde{Q}_\psi \hat{Q}_\psi d\varphi d\psi + \int_0^\phi \int_0^m \varphi^{-\lambda-1} b(\tilde{Q}) \tilde{Q}_{\varphi\psi} \hat{Q}_\psi d\varphi d\psi. \end{aligned} \quad (5.137)$$

Integrating by parts leads to

$$\begin{aligned} & \int_0^\phi \int_0^m \varphi^{-\lambda-1} h Q \tilde{Q}_{\varphi\psi} \tilde{Q}_\psi d\varphi d\psi \\ &= \frac{1}{2} \int_0^m \varphi^{-\lambda-1} h Q \tilde{Q}_\psi^2 d\psi \Big|_{\varphi=\phi} - \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} h Q)_\varphi \tilde{Q}_\psi^2 d\varphi d\psi. \end{aligned} \quad (5.138)$$

Summing up from (5.133) to (5.138) yields

$$0 = I_1 + I_2 + J_1 + J_2,$$

where

$$\begin{aligned} I_1 &= \frac{\lambda+1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-2} (\tilde{Q}_\varphi^2 + \hat{Q}_\varphi^2 - 2\tilde{Q}_\varphi \hat{Q}_\varphi) d\varphi d\psi \\ &= \frac{\lambda+1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2 d\varphi d\psi, \\ I_2 &= -\frac{1}{2} \int_0^\phi \int_0^m \left((\varphi^{-\lambda-1} b(\tilde{Q}))_\varphi \tilde{Q}_\psi^2 + (\varphi^{-\lambda-1} b(\hat{Q}))_\varphi \hat{Q}_\psi^2 - 2(\varphi^{-\lambda-1} b(\tilde{Q}))_\varphi \tilde{Q}_\psi \hat{Q}_\psi \right) d\varphi d\psi \\ & \quad - \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} h Q)_\varphi \tilde{Q}_\psi^2 d\varphi d\psi \\ & \quad - \int_0^\phi \int_0^m \varphi^{-\lambda-1} \left(b(\hat{Q}) \tilde{Q}_{\varphi\psi} \hat{Q}_\psi - b(\tilde{Q}) \tilde{Q}_{\varphi\psi} \hat{Q}_\psi + h Q \tilde{Q}_{\varphi\psi} \tilde{Q}_\psi \right) d\varphi d\psi \\ &= -\frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} b(\tilde{Q}))_\varphi Q_\psi^2 d\varphi d\psi - \frac{1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-1} h (\tilde{Q}_\psi + \hat{Q}_\psi) Q_\varphi Q_\psi d\varphi d\psi \\ & \quad - \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} h)_\varphi (\tilde{Q}_\psi + \hat{Q}_\psi) Q Q_\psi d\varphi d\psi - \int_0^\phi \int_0^m \varphi^{-\lambda-1} h \tilde{Q}_{\varphi\psi} Q Q_\psi d\varphi d\psi, \\ J_1 &= \frac{1}{2} \int_0^m \varphi^{-\lambda-1} (\tilde{Q}_\varphi^2 + \hat{Q}_\varphi^2 - 2\tilde{Q}_\varphi \hat{Q}_\varphi) d\psi \Big|_{\varphi=\phi} \\ & \quad + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} \left(b(\tilde{Q}) \tilde{Q}_\psi^2 - 2b(\tilde{Q}) \tilde{Q}_\psi \hat{Q}_\psi + b(\hat{Q}) \hat{Q}_\psi^2 + h Q \tilde{Q}_\psi^2 \right) d\psi \Big|_{\varphi=\phi} \end{aligned}$$

$$= \frac{1}{2} \int_0^m \varphi^{-\lambda-1} Q_\varphi^2 d\psi \Big|_{\varphi=\phi} + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} b(\tilde{Q}) Q_\psi^2 d\psi \Big|_{\varphi=\phi} + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} h(\tilde{Q}_\psi + \hat{Q}_\psi) Q Q_\psi d\psi \Big|_{\varphi=\phi}$$

and

$$\begin{aligned} J_2 &= - \int_0^\phi \varphi^{-\lambda-1} \left(b(\tilde{Q}) \tilde{Q}_\varphi \tilde{Q}_\psi - b(\tilde{Q}) \tilde{Q}_\psi \hat{Q}_\varphi + b(\hat{Q}) \hat{Q}_\varphi \hat{Q}_\psi - b(\hat{Q}) \tilde{Q}_\varphi \hat{Q}_\psi \right) d\varphi \Big|_{\psi=m} \\ &= - \int_0^\phi \varphi^{-\lambda-1} \left(b(\tilde{Q}) \tilde{Q}_\psi Q_\varphi - b(\hat{Q}) \hat{Q}_\psi Q_\varphi \right) d\varphi \Big|_{\psi=m} \\ &= - \int_0^\phi \varphi^{-\lambda-1} h \tilde{Q}_\psi Q Q_\varphi d\varphi \Big|_{\psi=m} - \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m} \\ &= - \frac{1}{2} \varphi^{-\lambda-1} h \tilde{Q}_\psi Q^2 \Big|_{(\varphi,\psi)=(\phi,m)} + \frac{1}{2} \int_0^\phi (\varphi^{-\lambda-1} h \tilde{Q}_\psi)_\varphi Q^2 d\varphi \Big|_{\psi=m} - \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\lambda+1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2 d\varphi d\psi - \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} b(\tilde{Q}))_\varphi Q_\psi^2 d\varphi d\psi \\ & \quad + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} Q_\varphi^2 d\psi \Big|_{\varphi=\phi} + \frac{1}{2} \int_0^m \varphi^{-\lambda-1} b(\tilde{Q}) Q_\psi^2 d\psi \Big|_{\varphi=\phi} - \frac{1}{2} \varphi^{-\lambda-1} h \tilde{Q}_\psi Q^2 \Big|_{(\varphi,\psi)=(\phi,m)} \\ &= \frac{1}{2} \int_0^\phi \int_0^m \varphi^{-\lambda-1} h(\tilde{Q}_\psi + \hat{Q}_\psi) Q_\varphi Q_\psi d\varphi d\psi + \frac{1}{2} \int_0^\phi \int_0^m (\varphi^{-\lambda-1} h)_\varphi (\tilde{Q}_\psi + \hat{Q}_\psi) Q Q_\psi d\varphi d\psi \\ & \quad + \int_0^\phi \int_0^m \varphi^{-\lambda-1} h \tilde{Q}_\varphi Q Q_\psi d\varphi d\psi - \frac{1}{2} \int_0^m \varphi^{-\lambda-1} h(\tilde{Q}_\psi + \hat{Q}_\psi) Q Q_\psi d\psi \Big|_{\varphi=\phi} \\ & \quad - \frac{1}{2} \int_0^\phi (\varphi^{-\lambda-1} h \tilde{Q}_\psi)_\varphi Q^2 d\varphi \Big|_{\psi=m} + \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m}. \end{aligned}$$

It follows from the regularity estimates of \tilde{Q} , \hat{Q} and h , the asymptotic behavior of \tilde{Q}_ψ and $\tilde{Q}_{\varphi\psi}$ on the wall near $\varphi = 0$ and the Hölder inequality that

$$\begin{aligned} & \int_0^\phi \int_0^m \left(\varphi^{-\lambda-2} Q_\varphi^2 + \varphi^{-3\lambda/2-3} Q_\psi^2 \right) d\varphi d\psi + \int_0^m \left(\varphi^{-\lambda-1} Q_\varphi^2 + \varphi^{-3\lambda/2-2} Q_\psi^2 \right) d\psi \Big|_{\varphi=\phi} \\ & \leq M \int_0^\phi \int_0^m \varphi^{-\lambda-3} |Q_\varphi Q_\psi| d\varphi d\psi + M \int_0^\phi \int_0^m \varphi^{-\lambda-4} |Q Q_\psi| d\varphi d\psi + M \int_0^\phi \int_0^m \varphi^{-5\lambda/4-7/2} |Q Q_\psi| d\varphi d\psi \\ & \quad + M \int_0^m \varphi^{-\lambda-3} |Q Q_\psi| d\psi \Big|_{\varphi=\phi} + M \int_0^\phi \varphi^{-\lambda-4} Q^2 d\varphi \Big|_{\psi=m} + \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m} \\ & \leq M \int_0^\phi \int_0^m \varphi^{-3\lambda/4-5/2} Q_\varphi^2 d\varphi d\psi + M \int_0^\phi \int_0^m \varphi^{-5\lambda/4-7/2} Q_\psi^2 d\varphi d\psi + M \int_0^\phi \int_0^m \varphi^{-5\lambda/4-7/2} Q^2 d\varphi d\psi \\ & \quad + M \int_0^m \varphi^{-\lambda-3} Q^2 d\psi \Big|_{\varphi=\phi} + M \int_0^m \varphi^{-\lambda-3} Q_\psi^2 d\psi \Big|_{\varphi=\phi} \\ & \quad + M \int_0^\phi \varphi^{-\lambda-4} Q^2 d\varphi \Big|_{\psi=m} + \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m}. \end{aligned} \tag{5.139}$$

To estimate the last term on the right side of (5.139), one substitutes (5.123) into this term, then integrates by parts and uses the Hölder inequality and the regularity estimates of \hat{Q} , e_1 and e_2 on the

wall near $\varphi = 0$ to get

$$\begin{aligned}
& \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m} \\
&= \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}(\varphi, m)) (e_1(\varphi) X(\varphi) + e_2(\varphi) Q(\varphi, m)) Q_\varphi(\varphi, m) d\varphi \\
&= \phi^{-\lambda-1} b(\hat{Q}(\phi, m)) e_1(\phi) X(\phi) Q(\phi, m) + \frac{1}{2} \phi^{-\lambda-1} b(\hat{Q}(\phi, m)) e_2(\phi) Q^2(\phi, m) \\
&\quad - \int_0^\phi (\varphi^{-\lambda-1} b(\hat{Q}(\varphi, m)) e_1(\varphi))' X(\varphi) Q(\varphi, m) d\varphi - \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}(\varphi, m)) e_1(\varphi) X'(\varphi) Q(\varphi, m) d\varphi \\
&\quad - \frac{1}{2} \int_0^\phi (\varphi^{-\lambda-1} b(\hat{Q}(\varphi, m)) e_2(\varphi))' Q^2(\varphi, m) d\varphi \\
&\leq M \phi^{-2} |X(\phi) Q(\phi, m)| + M \phi^{-\lambda-3} Q^2(\phi, m) + M \int_0^\phi \varphi^{-\lambda-1} |X(\varphi) Q(\varphi, m)| d\varphi \\
&\quad + M \int_0^\phi \varphi^{-2} |X'(\varphi) Q(\varphi, m)| d\varphi + M \int_0^\phi \varphi^{-\lambda-4} Q^2(\varphi, m) d\varphi \\
&\leq M \phi^{\lambda-1} X^2(\phi) + M \phi^{-\lambda-3} Q^2(\phi, m) + M \int_0^\phi \varphi^{-\lambda/2-1} X^2(\varphi) d\varphi + M \int_0^\phi \varphi^{-3\lambda/2-1} Q^2(\varphi, m) d\varphi \\
&\quad + M \int_0^\phi \varphi^\lambda |X'(\varphi)|^2 d\varphi + M \int_0^\phi \varphi^{-\lambda-4} Q^2(\varphi, m) d\varphi,
\end{aligned}$$

which, together with (5.130)–(5.132), yields

$$\begin{aligned}
& \int_0^\phi \varphi^{-\lambda-1} b(\hat{Q}) Q_\varphi Q_\psi d\varphi \Big|_{\psi=m} \\
&\leq M(\phi^{\lambda+2} + \phi^{\lambda/2+5}) \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2 d\varphi d\psi + M(\phi^{\lambda/2-1} + \phi^2) \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2 d\varphi d\psi \\
&\quad + M \phi^{\lambda/2-1} \int_0^m \varphi^{-3\lambda/2-2} Q_\psi^2 d\psi \Big|_{\varphi=\phi}. \tag{5.140}
\end{aligned}$$

Substituting (5.140) into (5.139) and using (5.131), we obtain

$$\begin{aligned}
& \int_0^\phi \int_0^m (\varphi^{-\lambda-2} Q_\varphi^2 + \varphi^{-3\lambda/2-3} Q_\psi^2) d\varphi d\psi + \int_0^m (\varphi^{-\lambda-1} Q_\varphi^2 + \varphi^{-3\lambda/2-2} Q_\psi^2) d\psi \Big|_{\varphi=\phi} \\
&\leq M \phi^{\lambda/4-1/2} \int_0^\phi \int_0^m \varphi^{-\lambda-2} Q_\varphi^2 d\varphi d\psi + M(\phi^{\lambda/4-1/2} + \phi^2) \int_0^\phi \int_0^m \varphi^{-3\lambda/2-3} Q_\psi^2 d\varphi d\psi \\
&\quad + M \phi^{\lambda/2-1} \int_0^m \varphi^{-3\lambda/2-2} Q_\psi^2 d\psi \Big|_{\varphi=\phi}.
\end{aligned}$$

Owing to $\lambda > 2$, one can conclude that for sufficiently small $\phi \in (0, \zeta_+]$,

$$Q_\varphi(\varphi, \psi) = 0, \quad Q_\psi(\varphi, \psi) = 0, \quad (\varphi, \psi) \in (0, \phi) \times (0, m).$$

Therefore,

$$\tilde{Q}(\varphi, \psi) = \hat{Q}(\varphi, \psi), \quad (\varphi, \psi) \in (0, \phi) \times (0, m).$$

□

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